ON SOJOURN OF BROWNIAN MOTION INSIDE MOVING BOUNDARIES

STÉPHANE SEURET AND XIAOCHUAN YANG

Abstract. We investigate the large scale structure of certain sojourn sets of one dimensional Brownian motion within two-sided moving boundaries. The macroscopic Hausdorff dimension and upper mass dimension of these sets are computed. We also give a uniform macroscopic dimension result for the Brownian level sets.

1. Introduction

This article is concerned with the sojourn properties of the one-dimensional Brownian motion $\{B_t : t \geq 0\}$ within some moving boundaries. More precisely, for an appropriate function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, consider the sets

$$E(\varphi) = \{t \geq 0 : |B_t| \leq \varphi(t)\}. \tag{1}$$

that we call the set of Brownian sojourn within two sided boundary $\varphi(\cdot)$.

Besides its obvious application in physics and finance, the understanding of these sets at different scales entails considerable information on the path properties of the Brownian motion. Two types of study are of particular interest.

• Geometric properties of $E(\varphi)$ near $t_0 = 0$. This corresponds to the regular behavior of a Brownian path near zero. Concretely, local asymptotics of the Brownian motion such as Khintchine's law of iterated logarithm can be described in terms of geometric properties of these sets around 0 with specific choices for φ. A natural question is under which condition on φ these sets admit an upper density with respect to Lebesgue measure (denoted by $|\cdot|$ throughout the paper), i.e.

$$\limsup_{s\to 0} \frac{|E(\varphi) \cap [0, s]|}{s} = c_\varphi > 0.$$

Uchiyama [20] treated the case $\varphi(t) = h(t)\sqrt{t}$ where h is taken from a whole class of correction functions (of logarithmic order), and he found the precise value of the constant c_φ.

• Geometric properties of $E(\varphi)$ at infinity. This is related to the long time behavior of the Brownian motion. As $\{B_t : t \geq 0\}$ behaves like a square root function at infinity (in expectation), the set $E(\varphi)$, when φ grows slower than the square root function, concerns the lower than normal growth of the Brownian motion. Uchiyama [20] established upper density bounds for $\varphi(t) = \sqrt{t}/h(t)$ with h belonging to a large class of correction functions with logarithmic order.

Let us mention briefly some related work. Consider

$$\widetilde{E}(\varphi) = \{t \geq 0 : |B_t| \geq \varphi(t)\}.$$

where φ grows like a square root function with a logarithmic order correction. The geometry of these sets around zero describes the local behavior of Brownian motion, whereas their geometry

Date: June 14, 2017.

Key words and phrases. 60G51, 28A80, 60G17: Macroscopic Hausdorff dimension, mass dimension, Brownian motion, sojourn sets.

around infinity describes the distribution of high peaks of the Brownian motion - we refer the interested reader to Strassen [18] and Uchiyama [20] for upper density results (around zero and infinity). We also mention the recent work of Khoshnevisan, Kim and Xiao [11] and Khoshnevisan and Xiao [12] who consider, among other things, the high peaks of symmetric stable Lévy processes. We refer the reader to [8, 16, 4, 5, 3] for other results on sojourn properties of stochastic processes.

Motivated by these studies, we focus on the asymptotic behavior around infinity of the sojourn sets of Brownian motion within moving boundaries with much lower than normal growth. For this, we introduce the sets

$$\forall\, \gamma \in [0,1/2], \quad E_\gamma := E(\varphi_\gamma) \text{ with } \varphi_\gamma(t) = t^\gamma,$$

and our goal is to estimate the size of E_γ for all values of γ. The "size" is expressed in terms of the notion of large scale dimensions developed by Barlow and Taylor [1, 2] in the late 80's, recently "refreshed" in the work of Xiao and Zheng [21], Georgiou *et al.* [9], Khoshnevisan *et al.* [11, 12].

The initial motivation of Barlow and Taylor was to define a notion of fractals in discrete spaces such as $\mathbb{Z}^d$. This permits to describe the size properties of models in physical statistics, such as the infinite connected component of a percolation process, the range of a transient random walk for instance. To this end, they introduced and investigated several notions of dimension describing different types of asymptotic behavior of a set around infinity. Each dimension corresponds to an analogue in large scales of a classical fractal dimension. Among these dimensions, we are going to use the "macroscopic Hausdorff dimension" and the "mass dimensions", which are respectively the analogs of the classical Hausdorff and Minkowski dimensions in large scales.

Let us state our main result. The macroscopic Hausdorff dimension of a set $E \subset \mathbb{R}^d$ is denoted by $\mathrm{Dim}_{\mathrm{H}} E$ and the upper mass dimension by $\mathrm{Dim}_{\mathrm{UM}} E$. Their definitions are recalled in Section 2.

Theorem 1. *Almost surely, for all $\gamma \in [0,1/2]$,*

$$\mathrm{Dim}_{\mathrm{H}} E_\gamma = \begin{cases} \frac{1}{2} & \text{if } \gamma \in [0,1/2) \\ 1 & \text{if } \gamma = 1/2 \end{cases} \tag{2}$$

$$\mathrm{Dim}_{\mathrm{UM}} E_\gamma = \frac{1}{2} + \gamma. \tag{3}$$

It is quite surprising that the macroscopic Hausdorff dimension of E_γ is constant for all $\gamma \in [0,1/2)$. This can be interpreted by the fact that, from the macroscopic Hausdorff dimension standpoint, B spends most of the time at its "boundary", i.e. the farthest possible from 0.

Further, one notices in the proof below that a.s. the Brownian zero set

$$\mathcal{Z} = \{t \geq 0 : B_t = 0\},$$

which is *a priori* thinner than E_0 (hence than all the E_γ's), also has macroscopic Hausdorff dimension 1/2. The local structure of $\mathcal{Z}$ is well understood since the works by Taylor and Wendel [19] and Perkins [15], who proved Hausdorff measure results for $\mathcal{Z}$ using local times. In particular, the classical Hausdorff dimension of $\mathcal{Z}$ is 1/2, a result already known in [7]. Our result gives the large scale structure of $\mathcal{Z}$ and might be compared with an interesting result by Khoshnevisan [10] who states that the zero set of a symmetric random walk $\{\xi_n : n \in \mathbb{N}\}$ in $\mathbb{Z}^1$ with finite variance

$$\{n \in \mathbb{N} : \xi_n = 0\}$$

has macroscopic Hausdorff dimension 1/2.

Although they have the same macroscopic Hausdorff dimension, the sets E_γ differ by their upper mass dimensions $\mathrm{Dim}_{\mathrm{UM}}$. Indeed, (3) indicates a multifractal behavior for $\mathrm{Dim}_{\mathrm{UM}} E_\gamma$, and also gives a natural example of sets for which the macroscopic Hausdorff dimension and the upper mass dimension differ.

Our contribution to the formula (2) is for all $\gamma \in [0,1/2)$. The case $\gamma = 1/2$ is deduced from Theorem 2 in [20] where Uchiyama obtained that a.s.

$$\limsup_{r\to+\infty} \frac{|E(1/2)\cap[0,r]|}{r} > 0.$$

This inequality entails that a.s. $\mathrm{Dim}_{\mathrm{H}} E(1/2) = 1$ thanks to the following fact proved in [11] : for any $E \subset \mathbb{R}$,

$$\Big(\limsup_{r\to+\infty} \frac{|E\cap[0,r]|}{r} > 0\Big) \Rightarrow \mathrm{Dim}_{\mathrm{H}} E = 1.$$

As B is recurrent, it does return inside the boundary φ_γ infinitely many times, for every $\gamma \in [0,1/2]$. In particular, the sets E_γ for $\gamma \in [0,1/2]$ are unbounded. Our result allows to quantify the recurrence and fluctuation properties of Brownian motion at large scales.

Through an equivalent definition for the macroscopic Hausdorff dimension, we also obtain a uniform dimension result for the level sets

$$\mathcal{Z}_x = \{t \geq 0 : B_t = x\}.$$

Theorem 2. *Almost surely, there exists at most one real number $x_0 \in \mathbb{R}$ such that for every $x \in \mathbb{R}\setminus\{x_0\}$, $\mathrm{Dim}_{\mathrm{H}}(\mathcal{Z}_x) = 1/2$.*

Theorem 2 should be compared with Perkins' uniform (classical Hausdorff) dimension result for these level sets [15]. Our method leaves the possibility of an exceptional point, that we were not able to dismiss; but we strongly believe that the result is globally uniform.

This paper is organized as follows. In Section 2, we recall the definition of large scale dimensions and establish some preliminary estimates on the probability that the Brownian motion belongs to a domain bounded by a moving boundary during some arbitrary time interval, on the Brownian local time increments around infinity, and on the hitting probability of subordinators. The first part of the main theorem (the macroscopic Hausdorff dimension) is proved in Section 3 and the second part (upper mass dimension) in Section 4. Theorem 2 is proved in Section 5.

2. Preliminaries

Throughout the paper, c, C are generic positive finite constants whose value may change from line to line. For two families of positive real numbers $(a(x))$ and $(b(x))$, the equation $a(x) \asymp b(x)$ means that the ratio $a(x)/b(x)$ is uniformly bounded from below and above by some positive finite constant independent of x. Also, $\mathbb{P}^x$ denotes the law of Brownian motion starting from $x \in \mathbb{R}$, and $\mathbb{E}^x$ denotes the expectation with respect to $\mathbb{P}^x$. For simplicity, we also write $\mathbb{P} = \mathbb{P}^0$ and $\mathbb{E} = \mathbb{E}^0$.

2.1. Macroscopic dimensions.

We adopt the notations in [11] that we recall now. We will use the notation $Q(x,r) = [x, x+r)$. The length of an interval $Q \subset \mathbb{R}$ is denoted by $s(Q)$.

Define the annuli $\forall\, n \geq 1, \mathcal{S}_n = [2^{n-1}, 2^n)$ and $\mathcal{S}_0 = [0,1)$. For any $\rho \geq 0$, any set $E \subset \mathbb{R}^+$, $n \in \mathbb{N}^*$, we introduce the quantity

$$\nu_\rho^n(E) = \inf\left\{ \sum_{i=1}^m \left(\frac{s(Q_i)}{2^n}\right)^\rho : E \cap \mathcal{S}_n \subset \bigcup_{i=1}^m Q_i \text{ with } s(Q_i) \geq 1 \text{ and } Q_i \subset \mathcal{S}_n \right\}. \tag{4}$$

Other gauge functions could be used instead of $x \mapsto x^\rho$, but we will not need them here.

Definition 1. *Let $E \subset \mathbb{R}^+$. The macroscopic Hausdorff dimension of E is defined as*

$$\mathrm{Dim_H} E = \inf \left\{ \rho \geq 0 : \sum_{n\geq 0} \nu_\rho^n(E) < +\infty \right\}. \tag{5}$$

The upper and lower mass dimension of E are defined as

$$\mathrm{Dim_{UM}} E = \limsup_{n\to+\infty} \frac{\ln(|E\cap[0,n]|)}{\ln n},$$
$$\mathrm{Dim_{LM}} E = \liminf_{n\to+\infty} \frac{\ln(|E\cap[0,n]|)}{\ln n}.$$

The macroscopic Hausdorff dimension of a set does not depend on any of its bounded subsets, since the series in (5) converges if and only if its tail series converges. Further, the covering intervals are chosen to have length larger than 1, which explains why the macroscopic Hausdorff dimension does not rely on the local structure of the underlying set. The same remarks apply to mass dimensions.

It is known [1, 2] that for any set $E \subset \mathbb{R}$,

$$\mathrm{Dim_H} E \leq \mathrm{Dim_{LM}} E \leq \mathrm{Dim_{UM}} E.$$

To bound $\mathrm{Dim_H} E$ from above, one usually exhibits an economic covering of E. To get the lower bound, the following lemma, which is an analog of the mass distribution principle, is useful.

Lemma 1. *Let $E \subset \mathcal{S}_n$. Let μ be a finite Borel measure on $\mathbb{R}$ with support included in E. Suppose that there exists finite positive constants C and ρ, such that for any interval $Q(x,r) \subset \mathcal{S}_n$ with $r \geq 1$, one has*

$$\mu(Q(x,r)) \leq C r^\rho.$$

Then

$$\nu_\rho^n(E) \geq C^{-1} 2^{-n\rho} \mu(\mathcal{S}_n).$$

Actually, there is more flexibility in the choice of covering intervals when we are only concerned with the value of the macroscopic dimension. Let us introduce, for every integer $n \geq 1$ and any set $E \subset \mathbb{R}^+$, the quantity

$$\widetilde{\nu}_\rho^n(E) = \inf \left\{ \sum_{i=1}^m \left(\frac{s(Q_{n,i})}{2^n} \right)^\rho : E \cap \mathcal{S}_n \subset \bigcup_{i=1}^m Q_{n,i} \text{ with } \frac{s(Q_{n,i})}{n^2} \in \mathbb{N}^* \text{ and } Q_{n,i} \subset \mathcal{S}_n \right\}. \tag{6}$$

The difference with ν_ρ^n is that coverings by sets of size that are multiple of n^2 are used, and this does not change the value of the macroscopic Hausdorff dimension, as stated by the following lemma.

Lemma 2. *For every set $E \subset \mathbb{R}^+$,*

$$\mathrm{Dim_H} E = \inf \left\{ \rho \geq 0 : \sum_{n\geq 0} \widetilde{\nu}_\rho^n(E) < +\infty \right\}. \tag{7}$$

Proof. Let us denote by $\widetilde{d}$ the value in the right hand-side of (7).

Obviously, for every $\rho \geq 0$, $\widetilde{\nu}_\rho^n(E) \geq \nu_\rho^n(E)$. Hence, recalling (5), $\mathrm{Dim_H} E \leq \widetilde{d}$. If $\mathrm{Dim_H}(E) = 1$, then (7) holds trivially. Assume thus $\mathrm{Dim_H}(E) < 1$.

Let $\rho' > \rho > \mathrm{Dim_H}(E)$, and fix $n \geq 1$. Choose $Q_n := \{Q_i\}_{i=1,\dots,m}$ as a finite family of intervals such that $E \cap \mathcal{S}_n \subset \bigcup_{i=1}^m Q_i$, with $s(Q_i) \geq 1$ and $\sum_{i=1}^m \left(\frac{s(Q_i)}{2^n}\right)^\rho \leq 2\nu_\rho^n(E)$.

Define the finite family of intervals $\widetilde{Q}_n := \{\widetilde{Q}_i\}_{i=1,\dots,m}$ as follows: $\widetilde{Q}_i$ is an interval containing Q_i, included in $\mathcal{S}_n$, whose length is the smallest possible multiple of n^2.

Observe that if $s(Q_i) \geq n^2$, $s(\widetilde{Q}_i) \leq 2s(Q_i)$, while if $1 \leq s(Q_i) < n^2$, $s(\widetilde{Q}_i) \leq n^2 s(Q_i)$.

By construction, $E \cap \mathcal{S}_n \subset \bigcup_{i=1}^m \widetilde{Q}_i$. In addition, since $\rho < \rho' < 1$,

$$\begin{aligned}
\sum_{i=1}^m \left(\frac{s(\widetilde{Q}_i)}{2^n}\right)^{\rho'} &= \sum_{i=1:s(Q_i)<n^2}^m \left(\frac{s(\widetilde{Q}_i)}{2^n}\right)^{\rho'} + \sum_{i=1:s(Q_i)\geq n^2}^m \left(\frac{s(\widetilde{Q}_i)}{2^n}\right)^{\rho'} \\
&\leq \sum_{i=1:s(Q_i)<n^2}^m n^2 \left(\frac{s(Q_i)}{2^n}\right)^{\rho'} + 2 \sum_{i=1:s(Q_i)\geq n^2}^m \left(\frac{s(Q_i)}{2^n}\right)^{\rho'} \\
&\leq \frac{n^{2+\rho'-\rho}}{2^{n(\rho'-\rho)}} \sum_{i=1:s(Q_i)<n^2}^m \left(\frac{s(Q_i)}{2^n}\right)^{\rho} + 2 \sum_{i=1:s(Q_i)\geq n^2}^m \left(\frac{s(Q_i)}{2^n}\right)^{\rho}.
\end{aligned}$$

When n becomes large, $\frac{n^{2+\rho'-\rho}}{2^{n(\rho'-\rho)}} \leq 1$, hence $\sum_{i=1}^m \left(\frac{s(\widetilde{Q}_i)}{2^n}\right)^{\rho'} \leq 2\sum_{i=1}^m \left(\frac{s(Q_i)}{2^n}\right)^{\rho}$.

One deduces that $\widetilde{\nu}^n_{\rho'}(E) \leq 2\nu^n_\rho(E)$. Since $\sum_{n\geq 1} \nu^n_\rho(E) < +\infty$, the series $\sum_{n\geq 1} \widetilde{\nu}^n_{\rho'}(E)$ also converges, and $\widetilde{d} \leq \rho'$. Letting ρ' tend to $\mathrm{Dim}_{\mathrm{H}}(E)$ yields the result. □

Remark 1. *The same argument shows that for any $C \geq 1$, using coverings of a set $E \cap \mathcal{S}_n$ by sets of size larger than C or than Cn^2 instead 1 or n^2 in the definitions (4) or (6), does not change the value of the macroscopic Hausdorff dimension of E.*

2.2. **Hitting probability estimates of Brownian motion inside the moving boundaries.** The following estimate is useful when looking for an appropriate covering of E_γ with respect to different large scale dimensions.

Lemma 3. *Consider an interval $Q(a,r)$ inside $\mathcal{S}_n$, i.e. $a \in \mathcal{S}_n$ and $a + r \leq 2^n$. For each $0 \leq \gamma < 1/2$, define the event*

$$\mathcal{A}(n,a,r,\gamma) = \{\exists\, t \in Q(a,r) : |B_t| \leq t^\gamma\}. \tag{8}$$

One has

$$\mathbb{P}\left(\mathcal{A}(n,a,r,\gamma)\right) \leq \frac{2}{\sqrt{\pi}} 2^{n(\gamma-1/2)} + \frac{4}{\sqrt{2\pi}} \left(\frac{r}{a}\right)^{1/2}.$$

Following a standard vocabulary, the event $\mathcal{A}(n,a,r,\gamma)$ describes the hitting probability of $\{B_t : t \geq 0\}$ inside the moving boundary. Lemma 3 states that, when $\frac{r}{a}$ is small, this hitting probability is controlled by γ, while it behaves like $\left(\frac{r}{a}\right)^{1/2}$ when $\frac{r}{a}$ becomes large. Basic properties of Brownian motion used in the proof of Lemma 3 can be found in [17] or [14].

Proof of Lemma 3 : One has

$$\begin{aligned}
\mathbb{P}(\mathcal{A}(n,a,r,\gamma)) &\leq \mathbb{P}\left(\inf_{t\in Q(a,r)} |B_t| \leq 2^{n\gamma}\right) \\
&= \mathbb{P}\left(|B_a| \leq 2^{n\gamma}\right) + \mathbb{P}\left(|B_a| > 2^{n\gamma}, \inf_{t\in Q(a,r)} |B_t| \leq 2^{n\gamma}\right) \\
&:= P_1 + P_2.
\end{aligned}$$

By the self-similarity of B and recalling that $a \in \mathcal{S}_n$, one obtains

$$P_1 = \mathbb{P}(|B_1| \leq a^{-1/2} 2^{n\gamma}) \leq \sqrt{\frac{2}{\pi}} a^{-1/2} 2^{n\gamma} \leq \frac{2}{\sqrt{\pi}} 2^{n(\gamma-1/2)}.$$

Using the symmetry of B, one gets

$$\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{t\in Q(a,r)} B_t \leq 2^{n\gamma}\right) = \mathbb{P}\left(B_a < -2^{n\gamma}, \sup_{t\in Q(a,r)} B_t \geq -2^{n\gamma}\right).$$

Thus,

$$P_2 \leq 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{t\in Q(a,r)} B_t \leq 2^{n\gamma}\right) = 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{t\in Q(a,r)} (B_t - B_a) \leq 2^{2n\gamma} - B_a\right).$$

Set $\tilde{B}_h = B_{a+h} - B_a$ which is a Brownian motion independent of B_a. Using successively the self-similarity, the symmetry and the Markov property yields

$$\begin{aligned}
P_2 &\leq 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{0\leq h\leq r} \tilde{B}_h \leq 2^{n\gamma} - B_a\right) \\
&= 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{0\leq h\leq 1} \tilde{B}_h \leq r^{-1/2}(2^{n\gamma} - B_a)\right) \\
&= 2\mathbb{P}\left(B_a > 2^{n\gamma}, \sup_{0\leq h\leq 1} \tilde{B}_h \geq r^{-1/2}(B_a - 2^{n\gamma})\right) \\
&= 2\int_{2^{n\gamma}}^{+\infty} \mathbb{P}\left(\sup_{0\leq h\leq 1} \tilde{B}_h \geq r^{-1/2}(x - 2^{n\gamma})\right) e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}} := I_1 + I_2,
\end{aligned}$$

where I_1 is the integral between $2^{n\gamma}$ and $2^{n\gamma} + r^{1/2}$, and I_2 is the other part of the integral. On one hand, bounding from above the probability inside the integral by 1, one obtains

$$I_1 \leq 2\int_{2^{n\gamma}}^{2^{n\gamma}+r^{1/2}} e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}} \leq \frac{2}{\sqrt{2\pi}}\left(\frac{r}{a}\right)^{1/2}.$$

On the other hand, one knows by applying the reflection principle to $\tilde{B}$ that $\sup_{0\leq h\leq 1} \tilde{B}_h$ has the same distribution as $|\tilde{B}_1|$. Hence, using the tail probability estimates of standard Gaussian variable, one has

$$\begin{aligned}
I_2 &\leq 4\int_{2^{n\gamma}+r^{1/2}}^{+\infty} \mathbb{P}\left(\tilde{B}_1 \geq r^{-1/2}(x - 2^{n\gamma})\right) e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}} \\
&\leq 4\int_{2^{n\gamma}+r^{1/2}}^{+\infty} \frac{1}{r^{-1/2}(x - 2^{n\gamma})\sqrt{2\pi}} e^{-\frac{(x-2^{n\gamma})^2}{2r}} e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}} \\
&\leq 4\frac{1}{\sqrt{2\pi}}\int_{2^{n\gamma}}^{+\infty} e^{-\frac{(x-2^{n\gamma})^2}{2r}} \frac{dx}{\sqrt{2\pi a}} = \frac{4}{\sqrt{2\pi}}\left(\frac{r}{a}\right)^{1/2}\int_{2^{n\gamma}}^{+\infty} e^{-\frac{(x-2^{n\gamma})^2}{2r}} \frac{dx}{\sqrt{2\pi r}} = \frac{2}{\sqrt{2\pi}}\left(\frac{r}{a}\right)^{1/2}.
\end{aligned}$$

Therefore, one has established that

$$P_2 \leq \frac{4}{\sqrt{2\pi}}\left(\frac{r}{a}\right)^{1/2}.$$

Combining the estimates ends the proof. □

2.3. **Brownian local times and 1/2-stable subordinator.** Let us recall very briefly the notion of local times. Let f be a non-negative continuous even function with integral one. For each $\varepsilon > 0$, set $f_{\varepsilon,x}(\cdot) = f((\cdot - x)/\varepsilon)/\varepsilon$. Define

$$L_t^x = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_0^t f_{\varepsilon,x}(B_s)\mathrm{d}s.$$

It is known [13, Lemma 2.4.1] that the convergence occurs uniformly on $(t, y) \in [0, T] \times [-M, M]$, $\mathbb{P}^y$ almost surely, for any $y \in \mathbb{R}$ and $T, M > 0$. There exists a version of the process $\{L^x_t; (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$ that is jointly continuous [13, Theorem 2.4.2]. We work with this version in the following. For each fixed x, the process $\{L^x_t : t \geq 0\}$ is called the Brownian local time at x. Below we write $L_t = L^0_t$ for simplicity.

Note that the non-decreasing continuous process $\{L_t : t \geq 0\}$ gives rise to a Radon measure whose support is included in the Brownian zero set $\mathcal{Z}$ [13, Remark 3.6.2]. We will use this measure in an analog of Lemma 1 to study the large scale dimension of $\mathcal{Z}$. Next lemma gives the asymptotic behavior at infinity of the local time increments, which is the continuous analog of Corollary 2.2 in [10].

Lemma 4. *One has $\mathbb{P}$-a.s.*

$$\limsup_{n\to\infty} n^{-1} \sup_{\substack{t\leq 2^n \\ t\in\mathbb{N}}} \sup_{\substack{2\leq h\leq 2^{n-2} \\ h\in\mathbb{N}}} \frac{L_{t+h} - L_t}{\sqrt{h/\log_2 h}} < +\infty.$$

Proof. Set $S_t = \sup_{0\leq s\leq t} B_s$. A famous theorem by Lévy [17, page 240] states that the processes $\{S_t; t \geq 0\}$ and $\{L_t; t \geq 0\}$ have the same law. Also, by the reflection principle, $L_t = |B_t|$ in distribution for any fixed t. Therefore, classical Gaussian tail probability estimates [13, page 192] yield that for all t and $x > 0$,

$$\mathbb{P}(L_t \geq x) = \mathbb{P}(|B_1| \geq x/\sqrt{t}) \leq e^{-x^2/(2t)}. \tag{9}$$

For each $t \geq 0$, define $T_t = \inf\{s \geq t : B_s = 0\}$ the first hitting time at zero of Brownian motion after time t. Since L only increases when the Brownian motion hits zero, one has for almost every sample path ω that, $L_{t+h} - L_t \leq L_{T_t+h} - L_{T_t} = L_h \circ \theta_{T_t}$ for any $t, h \geq 0$, see [17, page 402] for the equality. Here θ is the usual shift operator on the canonical Wiener space.

This, combined with (9) and the strong Markov property at the stopping time T_t, implies that for every $x > 0$,

$$\begin{aligned}
\mathbb{P}\left(\sup_{\substack{t\leq 2^n \\ t\in\mathbb{N}}} \sup_{\substack{2\leq h\leq 2^{n-2} \\ h\in\mathbb{N}}} \frac{L_{t+h} - L_t}{\sqrt{h/\log_2 h}} \geq x \right) &\leq 2^{2n-2} \sup_{\substack{t\leq 2^n \\ 2\leq h\leq 2^{n-2}}} \mathbb{P}\left(\frac{L_{t+h} - L_t}{\sqrt{h/\log_2 h}} \geq x \right) \\
&\leq 2^{2n-2} \sup_{2\leq h\leq 2^{n-2}} \mathbb{P}\left(L_h \geq x\sqrt{h/\log_2 h} \right) \\
&\leq 2^{2n} e^{-x^2/(2n)}
\end{aligned}$$

Taking $x = 4n$ in the above inequality, and summing over $n \geq 1$, the Borel-Cantelli lemma yields the conclusion. □

It is well known that the right continuous inverse of the Brownian local time at zero $\{\sigma_t : t \geq 0\}$ defined by $\sigma_t = \inf\{s \geq 0 : L_s > t\}$ is a $\frac{1}{2}$-stable subordinator. Also, the closure of the range of the associated subordinator coincides with $\mathcal{Z}$, see [6] or [13]. A subordinator $\{\sigma_t : t \geq 0\}$ is a Lévy processes with increasing sample paths. It is said to be 1/2-stable if the Laplace transform of σ_1 is $e^{-\Phi(\lambda)}$ with

$$\Phi(\lambda) = (2\lambda)^{1/2}, \text{ for all } \lambda > 0.$$

The renewal function $U(x)$ is the distribution function of the 0-potential measure of $\{\sigma_t : t \geq 0\}$, i.e.

$$U(x) = \mathbb{E}\left[\int_0^{+\infty} \mathbf{1}_{\sigma_t\leq x} dt \right].$$

The 0-potential measure of $\{\sigma_t : t \geq 0\}$, denoted by $U(dx)$, characterizes the law of σ in the sense that its Laplace transform is $1/\Phi(\lambda)$ for all $\lambda > 0$. Tauberian theorems provide the relation between the Laplace exponent Φ of a subordinator and its renewal function. More precisely, one has

$$U(x) \asymp \frac{1}{\Phi(1/x)}, \tag{10}$$

see Section 1.3 in [6]. For a general subordinator, one has the following hitting probability estimate in terms of its renewal function. It gives bounds for the probability that its range touches a deterministic set $A \subset \mathbb{R}$.

Lemma 5 ([6], Lemma 5.5)**.** *For every $0 < a < b$, one has*

$$\frac{U(b) - U(a)}{U(b-a)} \leq \mathbb{P}\Big(\mathcal{R} \cap [a,b] \neq \emptyset\Big) \leq \frac{U(2b-a) - U(a)}{U(b-a)},$$

where $\mathcal{R} = \sigma\big([0,+\infty)\big)$ is the range of $\{\sigma_t : t \geq 0\}$.

Next lemma gives a more precise hitting probability estimate for 1/2-stable subordinators.

Lemma 6. *Let $\{\sigma_t : t \geq 0\}$ be a $\frac{1}{2}$-stable subordinator and $Q(a,r) = [a, a+r)$ be a interval in some annulus $\mathcal{S}_n$. One has*

$$c\left(\frac{r}{a}\right)^{1/2} \leq \mathbb{P}\Big(\mathcal{R} \cap Q(a,r) \neq \emptyset\Big) \leq C\left(\frac{r}{a}\right)^{1/2}. \tag{11}$$

where c, C are independent of a, r and n.

Proof. First, applying (10) yields

$$U((a+r)-a) = U(r) \geq Cr^{1/2}. \tag{12}$$

On the other hand, by the definition of $U(x)$, one obtains

$$U(2(a+r)-a) - U(a) = \int_0^{+\infty} \mathbb{P}(\sigma_t \in [a, a+2r])dt = \int_0^{+\infty}\int_a^{a+2r} p_t(x)dx\,dt$$

where $p_t(x)$ is the density of σ_t, given by

$$p_t(x) = C\frac{t}{x^{3/2}}e^{-\frac{t^2}{2x}}$$

and C is the normalizing constant such that $p_t(x)$ is a probability measure. As $Q(a,r) \subset \mathcal{S}_n$, $r < a$ and one deduces that

$$\begin{aligned} U(a+2r) - U(a) &\leq C\int_0^{+\infty}\int_a^{a+2r} \frac{t}{a^{3/2}}e^{-\frac{t^2}{6a}}dx\,dt \\ &\leq C\frac{r}{a^{3/2}}\int_0^{+\infty} te^{-t^2/(6a)}dt = C\frac{r}{a^{1/2}}. \end{aligned} \tag{13}$$

The right inequality in (11) follows then by combining (13), (12) and Lemma 5 applied with $b = a + r$.

Similarly, one gets by (10)

$$U((a+r)-a) = U(r) \leq Cr^{1/2},$$

and by $r < a$

$$U(a+r) - U(a) = \int_0^{+\infty} \mathbb{P}(\sigma_t \in [a, a+r])\, dt = \int_0^{+\infty} \int_a^{a+r} p_t(x) dx\, dt.$$
$$\geq C \int_0^{+\infty} \int_a^{a+2r} \frac{t}{a^{3/2}} e^{-\frac{t^2}{2a}} dx\, dt \;=\; C \frac{r}{a^{1/2}}. \tag{14}$$

Applying Lemma 5 with $b = a + r$ and using the last inequalities, one gets the left inequality in (11). □

3. Proof of Theorem 1 : macroscopic Hausdorff dimension

In this section, we prove the dimension formula (2). As said in the introduction, by Uchiyama's upper density result [20], it is enough to compute $\text{Dim}_{\text{H}} E_\gamma$ for all $\gamma \in [0, 1/2)$.

Let $0 \leq \gamma < 1/2$ be fixed throughout this section. Due to the monotonicity in γ of the sets E_γ, and the fact that the zero set of Brownian motion $\mathcal{Z} = \{t \geq 0 : B_t = 0\}$ is included in $E(0)$, we divide the proof of (2) into two parts :

$$\text{Dim}_{\text{H}} E_\gamma \leq \frac{1}{2} \quad \text{and} \quad \text{Dim}_{\text{H}} \mathcal{Z} \geq \frac{1}{2}.$$

Let us start with the upper bound. Let $\gamma \in [0, 1/2)$ and $\rho > 1/2$ be fixed. Set

$$x_{n,i} = 2^{n-1} + i 2^{2n\gamma} \text{ for } i \in \{0, \dots, \lfloor 2^{n-1}/2^{2n\gamma} \rfloor\}.$$

Consider the intervals $Q(x_{n,i}, 2^{2n\gamma})$ which form a partition of $\mathcal{S}_n$. Note that $E_\gamma \cap Q(x_{n,i}, 2^{2n\gamma}) = \emptyset$ outside of the event $\mathcal{A}(n, x_{n,i}, 2^{2n\gamma}, \gamma)$ (see its definition in Lemma 3). Thus,

$$\nu_\rho^n(E_\gamma) \leq \sum_{i=0}^{\lfloor 2^{(1-2\gamma)n-1} \rfloor} \left(\frac{2^{2n\gamma}}{2^n} \right)^\rho \mathbf{1}_{\mathcal{A}(n, x_{n,i}, 2^{2n\gamma}, \gamma)}.$$

By choosing the side length $2^{2n\gamma}$, one observes that the two terms in Lemma 3 are of the same order. Taking expectation in the above inequality, one obtains by Lemma 3 that there exists a positive finite constant C such that for all $n \in \mathbb{N}^*$

$$\begin{aligned} \mathbb{E}[\nu_\rho^n(E_\gamma)] &\leq 2^{(1-2\gamma)(1-\rho)n} \cdot C 2^{n(\gamma - 1/2)} \\ &= C 2^{(1/2-\gamma)(1-2\rho)n}. \end{aligned}$$

Thus, the Fubini Theorem entails $\mathbb{E}[\sum_{n \in \mathbb{N}^*} \nu_\rho^n(E_\gamma)] < +\infty$. This proves that $\text{Dim}_{\text{H}} E_\gamma \leq \rho$ almost surely. Letting $\rho \to 1/2$ yields the upper bound.

Next we move to the lower bound $\text{Dim}_{\text{H}} \mathcal{Z} \geq 1/2$. To this end, we follow the idea of Khoshnevisan [10, page 581].

Let $g(\varepsilon) = \sqrt{\varepsilon \log_2(1/\varepsilon)}$ for $0 < \varepsilon \leq 1/2$. Let $C > 0$ be an upper bound for the limsup in Lemma 4. There exists a random integer $n_0(\omega)$ such that for all $n \geq n_0(\omega)$,

$$\sup_{\substack{t \leq 2^n \\ t \in \mathbb{N}}} \sup_{\substack{2 \leq h \leq 2^{n-2} \\ h \in \mathbb{N}}} \frac{L_{t+h} - L_t}{\sqrt{h / \log_2 h}} \leq 2Cn. \tag{15}$$

Consider any finite sequence of intervals $\{Q_{n,i} = [a_{n,i}, b_{n,i}]\}_{i=1,\dots,m}$ with integer endpoints and with sidelength $2 \leq s(Q_{n,i}) \leq 2^{n-2}$, that form a covering of $\mathcal{Z} \cap \mathcal{S}_n$.

Using (15), one obtains

$$g\left(\frac{s(Q_{n,i})}{2^n}\right)=\sqrt{\frac{s(Q_{n,i})}{2^n}\log_2\frac{2^n}{s(Q_{n,i})}}=2^{-n/2}\sqrt{\frac{s(Q_{n_i})}{\log_2 s(Q_{n,i})}\left(n-\log_2 s(Q_{n_i})\right)\log_2 s(Q_{n,i})}$$
$$\geq (n-1)^{1/2}2^{-n/2}\frac{L_{b_{n,i}}-L_{a_{n,i}}}{2Cn}\geq\frac{n^{-1/2}2^{-n/2}}{4C}(L_{b_{n,i}}-L_{a_{n,i}}).$$

where we used the elementary inequality $\sqrt{x(n-x)}\geq\sqrt{n-1}$ for all $1\leq x\leq n-1$. Therefore, a.s. for all $n\geq n_0(\omega)$ and any covering of $\mathcal{Z}\cap\mathcal{S}_n$ by a finite family of intervals $(Q_{n,i})_{i=1,\dots,m}$,

$$\sum_{i=1}^m g\left(\frac{s(Q_{n,i})}{2^n}\right)\geq\frac{L_{2^n}-L_{2^{n-1}}}{4C\sqrt{n2^n}}.$$

The desired lower bound for $\mathrm{Dim}_{\mathrm{H}}\mathcal{Z}$ follows if we can show that a.s.

$$F_N=\sum_{n=1}^N\frac{L_{2^n}-L_{2^{n-1}}}{\sqrt{n2^n}} \tag{16}$$

diverges as $N\to+\infty$. Indeed, this would demonstrate that the sum (5) diverges for every $\rho<1/2$ and for any choice of covering.

To this end, let us use a second moment argument. As for any fixed t, $L_t=|B_t|$ in distribution, one knows that

$$\mathbb{E}[L_t]=c\sqrt{t}\text{ and }\mathbb{E}[(L_t)^2]=t. \tag{17}$$

An Abel summation manipulation gives

$$F_N=\frac{L_{2^N}}{\sqrt{N2^N}}-\frac{L_2}{\sqrt{2}}+\sum_{n=2}^N L_{2^n}\left(\frac{1}{\sqrt{n2^n}}-\frac{1}{\sqrt{(n+1)2^{n+1}}}\right).$$

Thus,

$$\mathbb{E}[F_N]\asymp\sqrt{N}.$$

By Cauchy-Schwarz inequality, $\mathbb{E}[F_N^2]\geq cN$. To deduce an upper bound for $\mathbb{E}[F_N^2]$, we come back to the expression (16). Recall that T_t is the first time the Brownian motion hits zero after time t and $L_{t+h}-L_t\leq L_{T_t+h}-L_{T_t}=L_h\circ\theta_{T_t}$. By the strong Markov property applied at the stopping times $T_{2^{n-1}}$, $T_{2^{\ell-1}}$, $T_{2^{j-1}}$ (for the integers j and ℓ defined below), one has using (17)

$$\mathbb{E}[F_N^2]=\sum_{n=1}^N\frac{\mathbb{E}[(L_{2^n}-L_{2^{n-1}})^2]}{n2^n}+2\sum_{\ell=1}^N\sum_{j=1}^{\ell-1}\frac{\mathbb{E}[(L_{2^j}-L_{2^{j-1}})(L_{2^\ell}-L_{2^{\ell-1}})]}{\sqrt{j2^j\ell2^\ell}}$$
$$\leq\sum_{n=1}^N\frac{1}{n}+2c\sum_{\ell=1}^N\sum_{j=1}^{\ell-1}(j\ell)^{-1/2}\leq 3cN.$$

Therefore, applying Paley-Zygmund inequality yields that there are constants $c,c'>0$ for which for every $N\geq1$,

$$\mathbb{P}(F_N\geq c\sqrt{N})\geq c'.$$

This proves $\mathbb{P}(F_\infty=\infty)>0$. By Kolmogorov's zero-one law, $F_\infty=\infty$ almost surely, which completes the proof.

4. Proof of Theorem 1 : upper mass dimension

In this section, we prove that almost surely, for all $\gamma \in [0, 1/2]$,

$$\mathrm{Dim}_{\mathrm{UM}} E_\gamma := \limsup_{n \to +\infty} \frac{\ln |E_\gamma \cap [0, n]|}{\ln n} = \frac{1}{2} + \gamma.$$

We are going to show that the sojourn time $|E_\gamma \cap \mathcal{S}_n|$ is larger than $2^{n(1/2+\gamma)}$ for infinitely many integers n, but at the same time never exceeds $2^{n(1/2+\gamma+\varepsilon)}$ for all large n, where $\varepsilon > 0$ can be chosen arbitrarily small. As the underlying sequence of sojourn time are dependent random variables, the following strong law of large numbers for dependent events, proved in Xiao and Zheng [21], is needed.

Lemma 7. [21, Lemma 2.9] *Suppose that $\{\mathcal{A}_k\}_{k \geq 1}$ and $\{\mathcal{D}_k\}_{k \geq 1}$ are two sequences of events adapted to the same filtration $\{\mathcal{F}_k\}_{k \geq 1}$ and are such that for some positive constants p, a and δ*

$$\forall\, k \geq 1, \;\; \mathbb{P}(\mathcal{A}_{k+1} | \mathcal{F}_k) \geq p \text{ on the event } \mathcal{D}_k, \;\text{ and } \mathbb{P}(\mathcal{D}_k^c) \leq a e^{-\delta k}.$$

Then there exists $\varepsilon > 0$ such that almost surely,

$$\liminf_{n \to +\infty} \frac{\sum_{k=1}^n \mathbf{1}_{\mathcal{A}_k}}{n} \geq \varepsilon.$$

Let us first prove the upper bound $\mathrm{Dim}_{\mathrm{UM}} E_\gamma \leq \frac{1}{2} + \gamma$. For each annulus $\mathcal{S}_n$, consider the unit intervals $Q(k, 1)$ with left endpoint $k \in \mathcal{S}_n$.

Recall the definition of the events $\mathcal{A}(p, k, 1, \gamma)$ in Lemma 3. We observe that the union of those intervals $Q(k, 1)$ inside each $\mathcal{S}_p$ $(0 \leq p \leq n)$ satisfying $\mathbf{1}_{\mathcal{A}(p,k,1,\gamma)} = 1$ forms a covering of $E_\gamma \cap [0, 2^n]$. This implies that

$$|E_\gamma \cap [0, 2^n]| \leq \sum_{p=0}^{n} \sum_{k=2^{p-1}}^{2^p - 1} \mathbf{1}_{\mathcal{A}(p,k,1,\gamma)}.$$

Taking expectation, one obtains by Lemma 3 that for some finite positive constant C,

$$\mathbb{E}[|E_\gamma \cap [0, 2^n]|] \leq C \sum_{p=0}^{n} 2^p 2^{p(\gamma - 1/2)} = C 2^{n(\gamma + 1/2)}.$$

For $\rho > 1/2 + \gamma$, the first moment Markov inequality yields that for all $n \in \mathbb{N}^*$,

$$\mathbb{P}\left(|E_\gamma \cap [0, 2^n]| \geq 2^{n\rho}\right) \leq C^{-1} 2^{-n(\rho - \gamma - 1/2)}$$

which is the general term of a convergent series. Thus, an application of Borel-Cantelli Lemma gives almost surely, for all n large enough,

$$|E_\gamma \cap [0, 2^n]| < 2^{n\rho}.$$

Hence, for all m large enough, denote by n the unique integer such that $m \in \mathcal{S}_n$, one gets

$$\frac{\ln |E_\gamma \cap [0, m]|}{\ln m} \leq \frac{\ln |E_\gamma \cap [0, 2^n]|}{\ln 2^{n-1}} \leq \frac{n}{n-1} \rho$$

Taking $\limsup$ entails $\mathrm{Dim}_{\mathrm{UM}} E_\gamma \leq \rho$, almost surely. The desired upper bound follows by letting ρ tend to $\gamma + 1/2$.

Now we prove the lower bound $\mathrm{Dim}_{\mathrm{UM}} E_\gamma \geq \frac{1}{2} + \gamma$. Let us fix some notations before moving forward.

Define the super-exponential increasing sequence of integers

$$\begin{cases} n_0 = 1, \\ n_{k+1} = 2^{n_k} \text{ for all } k \in \mathbb{N}^*. \end{cases}$$

Set

$$\widehat{\mathcal{S}}_n = \mathcal{S}_n \cup \mathcal{S}_{n+1} \cup \mathcal{S}_{n+2}.$$

We introduce two sequences of events for $k \geq 1$

$$\mathcal{A}_k = \left\{ |E_\gamma \cap \widehat{\mathcal{S}}_{n_k}| \geq K 2^{n_k(\gamma+1/2)} \right\} \quad \text{and} \quad \mathcal{D}_k = \left\{ |B_{2^{n_k+2}}| \leq n_k \cdot 2^{(n_k+2)/2} \right\},$$

recalling that 2^{n_k+2} is the right endpoint of $\widehat{\mathcal{S}}_{n_k}$. Here $K \geq 1$ is a universal constant which will be chosen later.

One observes that when $\mathcal{A}_k$ is realized, $|E_\gamma \cap \mathcal{S}_n| \geq K 2^{n(\gamma+1/2)}/3$ for some $n \in \{n_k, n_k+1, n_k+2\}$. So if $\mathcal{A}_k$ is realized for infinitely many k's, $\frac{\ln |E_\gamma \cap \mathcal{S}_n|}{\ln 2^n} \geq \gamma + 1/2$ for infinitely many integers n, from which the result follows.

To this end, we first check that the events $\{\mathcal{A}_k\}_{k \geq 1}$ and $\{\mathcal{D}_k\}_{k\geq 1}$ verify the conditions of Lemma 7, and then apply this lemma.

Let $\mathcal{F}_k = \sigma(B_s : s \leq 2^{n_k+2})$. The condition on $\{\mathcal{D}_k\}_{k\geq1}$ is easy to check. Indeed, by the self-similarity of Brownian motion, the tail probability estimate of a Gaussian variable, and the rapid increasing rate of $\{n_k\}$, one gets that

$$\mathbb{P}(\mathcal{D}_k^c) = \mathbb{P}(|B_1| > n_k) \leq \frac{2}{n_k \sqrt{2\pi}} e^{-n_k^2/2} \leq e^{-k}.$$

It remains to check the condition on $\{\mathcal{A}_k\}_{k\geq1}$. Applying the Markov property of $\{B_t : t \geq 0\}$ at time 2^{n_k+2} yields

$$\mathbb{P}(\mathcal{A}_{k+1} | \mathcal{F}_k) = g(B_{2^{n_k+2}}),$$

where

$$\begin{aligned} g(x) &= \mathbb{P}^x \left(\int_{2^{n_{k+1}-1} - 2^{n_k+2}}^{2^{n_{k+1}+2} - 2^{n_k+2}} \mathbf{1}_{|B_t| \leq (t + 2^{n_k+2})^\gamma} dt \geq K 2^{n_{k+1}(1/2+\gamma)} \right) \\ &\geq \mathbb{P}^x \left(\int_{2^{n_{k+1}-1}}^{2 \cdot 2^{n_{k+1}}} \mathbf{1}_{|B_t| \leq t^\gamma} dt \geq K 2^{n_{k+1}(1/2+\gamma)} \right). \end{aligned}$$

Define the x-level set of the Brownian motion

$$\mathcal{Z}_x = \{t \geq 0 : B_t = x\}$$

and the stopping time

$$\tau_{k+1} = \inf\{t \in \mathcal{S}_{n_{k+1}} : B_t = x\}.$$

Since $[\tau_{k+1}, \tau_{k+1} + 2^{n_{k+1}}] \subset [2^{n_{k+1}-1}, 2 \cdot 2^{n_{k+1}}]$ under the event $\{\mathcal{Z}_x \cap \mathcal{S}_{n_{k+1}} \neq \emptyset\}$, one has

$$g(x) \geq \mathbb{P}^x \left(\mathcal{Z}_x \cap \mathcal{S}_{n_{k+1}} \neq \emptyset, \int_{\tau_{k+1}}^{\tau_{k+1} + 2^{n_{k+1}}} \mathbf{1}_{|B_t| \leq t^\gamma} dt \geq K 2^{n_{k+1}(1/2+\gamma)} \right).$$

By the strong Markov property of Brownian motion at τ_{k+1},

$$g(x) \geq \mathbb{P}^x \Big(\mathcal{Z}_x \cap \mathcal{S}_{n_{k+1}} \neq \emptyset \Big) \times \mathbb{P}^x \left(\int_0^{2^{n_{k+1}}} \mathbf{1}_{|B_t| \leq 2^{(n_{k+1}-1)\gamma}} dt \geq K 2^{n_{k+1}(1/2+\gamma)} \right) := P_k^1(x) P_k^2(x).$$

We estimate separately the two terms in the last product. Observe that $\mathcal{Z}_x$ under $\mathbb{P}^x$ coincides with the range of a 1/2-stable subordinator. So, the hitting probability estimate given by Lemma 6 yields that

$$P_k^1(x) \geq C \tag{18}$$

uniformly for all k. Finally, one observes that uniformly for all $x \in [-n_k \cdot 2^{(n_k+2)/2}, n_k \cdot 2^{(n_k+2)/2}]$,

$$\begin{aligned} P_k^2(x) &\geq \mathbb{P}\left(\int_0^{2^{n_{k+1}}} \mathbf{1}_{|B_t| \leq 2^{(n_{k+1}-1)\gamma} - x} dt \geq K 2^{n_{k+1}(1/2+\gamma)}\right) \\ &\geq \mathbb{P}\left(\int_0^{2^{n_{k+1}}} \mathbf{1}_{|B_t| \leq 2^{(n_{k+1}-1)\gamma - 1}} dt \geq K 2^{n_{k+1}(1/2+\gamma)}\right) := p_k \end{aligned} \tag{19}$$

The sequence $\{p_k\}_{k\geq 1}$ is uniformly controlled from below by the following lemma.

Lemma 8. *For $k \geq 1$, let*

$$M_k = \int_{2^{n_{k+1}-1}}^{2^{n_{k+1}}} \mathbf{1}_{|B_t| \leq 2^{(n_{k+1}-1)\gamma}/2} \, dt.$$

There exist positive constants C_1, C_1', C_2 such that for all $k \geq 1$

$$C_1 2^{n_{k+1}(\gamma+1/2)} \leq \mathbb{E}[M_k] \leq C_1' 2^{n_{k+1}(\gamma+1/2)}$$

and

$$\mathbb{E}[M_k^2] \leq C_2 2^{n_{k+1}(1+2\gamma)}.$$

Proof. An application of Fubini theorem and the self-similarity of $\{B_t : t \geq 0\}$ yields the first moment estimate. For the second moment, one has

$$\begin{aligned} \mathbb{E}[M_k^2] &= 2\int_{2^{n_{k+1}-1}}^{2^{n_{k+1}}} \int_t^{2^{n_{k+1}}} \mathbb{P}(|B_t| \leq 2^{(n_{k+1}-1)\gamma-1}, |B_s| \leq 2^{(n_{k+1}-1)\gamma-1}) ds\, dt \\ &\leq 2\int_{2^{n_{k+1}-1}}^{2^{n_{k+1}}} \int_0^{2^{n_{k+1}}} \mathbb{P}(|B_t| \leq 2^{(n_{k+1}-1)\gamma-1}) \mathbb{P}(|B_u| \leq 2^{(n_{k+1}-1)\gamma}) du\, dt \\ &\leq 2\int_{2^{n_{k+1}-1}}^{2^{n_{k+1}}} \mathbb{P}(|B_t| \leq 2^{(n_{k+1}-2)\gamma}) dt \left(1 + \sum_{i=1}^{n_{k+1}} \int_{2^{i-1}}^{2^i} \mathbb{P}(|B_u| \leq 2^{(n_{k+1}-1)\gamma}) du\right). \end{aligned}$$

Applying the first moment estimate yields the results. □

Applying Paley-Zygmund inequality to p_k with $K = C_1/2$, one obtains that for all k,

$$p_k \geq \frac{\mathbb{E}[M_k]^2}{4\mathbb{E}[M_k^2]} = \frac{C_1}{4C_2} > 0.$$

Combining this, (18) and (19), one deduces that for all k, on the event $\mathcal{D}_k$,

$$\mathbb{P}(\mathcal{A}_{k+1}|\mathcal{F}_k) = g(B_{2^{n_k+2}}) \geq C\frac{C_1}{4C_2} > 0.$$

Applying Lemma 7, there exists $\varepsilon > 0$ such that almost surely,

$$\liminf_{n\to+\infty} \frac{\sum_{k=1}^n \mathbf{1}_{\mathcal{A}_k}}{n} \geq \varepsilon.$$

In particular, $\mathcal{A}_k$ is realized for infinitely many k's. As explained before, this completes the proof for the lower bound.

5. Proof of Theorem 2 : dimension of all level sets

In the preceding sections, we have proved that almost surely, $\mathrm{Dim}_{\mathrm{H}}(\mathcal{Z}_0)=1/2$. We are going to prove that almost surely, for every $x\in\mathbb{R}$ except maybe one point, $\mathrm{Dim}_{\mathrm{H}}(\mathcal{Z}_x)=1/2$.

For this, we start with the following easy lemma. For any function $f:\mathbb{R}^+\to\mathbb{R}$ and any interval $I\subset\mathbb{R}^+$, the oscillation of f on I is written

$$\mathrm{Osc}_I(f)=\sup_I f-\inf_I f.$$

Lemma 9. *Let $\{B_t : t\geq 0\}$ be a Brownian motion. With probability one, there exists an integer $N\geq 1$ such that for every $n\geq N$, for any integer $\ell\in[2^n,2^{n+1}-n^{3/2}]$, there exists at least one interval among the consecutive intervals $\{I_k:=[\ell+k,\ell+k+1]\}_{k=0,\dots,n^{3/2}-1}$ such that the oscillation of B on I_k is larger than $\log\log n$.*

Proof. Fix $n\geq 1$. For any integer interval $[k,k+1]\subset\mathcal{S}_n$, let us denote $\tilde{p}_n=\mathbb{P}(\mathrm{Osc}_{[k,k+1]}B\geq \log\log n)>0$. A standard estimate shows that $\tilde{p}_n=C_ne^{-(\log\log n)^2/2}/\log\log n$, where C_n is a constant uniformly bounded by above and below with respect to n.

The Markov property implies that the oscillation of Brownian motion on non-overlapping unit intervals are independent and identically distributed. Therefore, for any sequence of $\lfloor n^{3/2}/2\rfloor$ consecutive intervals I_1, ..., $I_{\lfloor n^{3/2}/2\rfloor}$ of length 1, the probability that simultaneously the oscillation of $\{B_t : t\geq 0\}$ on each of these intervals is less than $\log\log n$ is $p_n:=(1-\tilde{p}_n)^{\lfloor n^{3/2}/2\rfloor}$. One sees that $p_n\sim e^{-C_nn^{3/2}e^{-(\log\log n)^2/2}/\log\log n}$, hence goes fast to 0 ($C_n$ is another constant, still bounded away from 0 and ∞).

There are $C2^n/\lfloor n^{3/2}/2\rfloor$ disjoint sequences of consecutive $\lfloor n^{3/2}/2\rfloor$ integer intervals included in $\mathcal{S}_n$. Hence, by independence, the probability that there is at least one such sequence of $\lfloor n^{3/2}/2\rfloor$ consecutive intervals such that the oscillation of $\{B_t : t\geq 0\}$ on all these intervals is less than $\log\log n$ is

$$\hat{p}_n:=1-(1-p_n)^{C2^n/\lfloor n^{3/2}/2\rfloor},\tag{20}$$

which tends exponentially fast to 1. The Borel-Cantelli lemma yields that there exists some (random) integer N such that for every $n\geq N$, there is always an interval I of length 1 in the $\lfloor n^{3/2}/2\rfloor$ consecutive intervals such that the oscillation of $\{B_t : t\geq 0\}$ on I is larger than $\log\log n$.

As a conclusion, with probability one, every interval of length $n^{3/2}$ (which contains necessarily $\lfloor n^{3/2}/2\rfloor$ consecutive intervals above) in $\mathcal{S}_n$ contains a subinterval of length 1 on which the oscillation of $\{B_t : t\geq 0\}$ is larger than $\log\log n$. □

We now prove that the macroscopic Hausdorff dimension is 1/2 over a semi-infinite interval. Recall that $\mathcal{Z}_x=\{t\geq 0 : B_t=x\}$.

Lemma 10. *For every $x_0\in\mathbb{R}$, almost surely, $\mathrm{Dim}_{\mathrm{H}}\mathcal{Z}_x=1/2$ either for all $x\geq x_0$ or for all $x\leq x_0$.*

Proof. We prove the Lemma for $x_0=0$. One knows that almost surely, $\mathrm{Dim}_{\mathrm{H}}\mathcal{Z}_0=1/2$.

First, Lemma 9 gives an integer N such that the conclusions of this lemma hold for $n\geq N$.

Let $Q_n=(Q_{n,i})_{i=1,\dots,m_n}$ be a finite family of intervals forming a covering of $\mathcal{Z}_0\cap\mathcal{S}_n$ by intervals of diameter multiple of n^2.

Each $Q_{n,i}$ can be split into a finite number of contiguous intervals Q of length n^2.

Consider one of these intervals Q, and enlarge it by adding on its right side an interval of size n^2. Call $\widetilde{Q}=Q+[0,n^2]$ the obtained interval.

If Q intersects $\mathcal{Z}_0$, by Lemma 9, when n is large, there exists an interval $I \subset \widetilde{Q}$ of length 1 on which the oscillation of B is greater than $\log\log n$. Hence, (at least) one of the two intervals $[0, \log\log n/2]$ or $[-\log\log n/2, 0]$ is necessarily included in $B_{\widetilde{Q}}$ (the image of $\widetilde{Q}$ by $\{B_t : t \geq 0\}$). So $\widetilde{Q} \cap \mathcal{Z}_x \neq \emptyset$, simultaneously for all $x \in [0, \log\log n/2]$ or for all $x \in [-\log\log n/2, 0]$.

We introduce the sets

$$Q^+_{n,i} = \bigcup_{Q \in Q_{n,i} :\, Q \cap \mathcal{Z}_0 \neq \emptyset,\, [0, \log\log n/2] \subset B_{\widetilde{Q}}} \widetilde{Q} \quad \text{and} \quad Q^-_{n,i} = \bigcup_{Q \in Q_{n,i} :\, Q \cap \mathcal{Z}_0 \neq \emptyset,\, [-\log\log n/2, 0] \subset B_{\widetilde{Q}}} \widetilde{Q}. \tag{21}$$

Observe that the two sets $Q^+_{n,i}$ and $Q^-_{n,i}$ are union of intervals, and that these unions may intersect each other.

Remark 2. *Notice that in this situation, for every $x \in [0, \log\log n/2]$ (resp. $x \in [-\log\log n/2, 0]$), if R_n is a covering of $\mathcal{Z}_x \cap Q_{n,i}$ by intervals I of size that are multiple of n^2, then $\bigcup_{I \in R_n} (I + [-n^2, n^2])$ necessarily contains $Q^+_{n,i}$ (resp. $Q^-_{n,i}$).*

Finally we set

$$Q^+_n = \bigcup_{i=1}^{m_n} Q^+_{n,i} \quad \text{and} \quad Q^-_n = \bigcup_{i=1}^{m_n} Q^-_{n,i}. \tag{22}$$

Let $\rho < 1/2$. By definition, almost surely, $\sum_{n \geq 0} \widetilde{\nu}^n_\rho(\mathcal{Z}_0) = +\infty$.

In order to simplify the notation, we denote for any family $Q_n = \{Q_{n,i}\}_{i=1,\ldots,m_n}$ of intervals included in $\mathcal{S}_n$

$$s_\rho(Q_n) := \sum_{i=1}^{m_n} \left(\frac{s(Q_{n,i})}{2^n} \right)^\rho. \tag{23}$$

By abuse of notation, we use $s(E)$ to denote the diameter of a set E even when E is not an interval.

Assume that for every integer $n \geq 1$, the family $Q_n = (Q_{n,i})_{i=1,\ldots,m_n}$ is a covering of $\mathcal{S}_n \cap \mathcal{Z}_0$ by intervals of length multiple of n^2 and in addition, that Q_n is one of the families that minimize the sum $s_\rho(Q_n)$ in (6), so that

$$\sum_{n \geq 0} s_\rho(Q_n) = +\infty. \tag{24}$$

It is obvious from the construction that $\mathcal{Z}_0 \cap Q_{n,i} \subset Q^-_{n,i} \cup Q^+_{n,i}$.

In addition, it is not possible to have $s(Q^-_{n,i})^\rho + s(Q^+_{n,i})^\rho < s(Q_{n,i})^\rho$, otherwise the family Q_n would not be a minimizer. Hence $\max(s(Q^-_{n,i})^\rho, s(Q^+_{n,i})^\rho) \geq s(Q_{n,i})^\rho/2$.

For the same reason, it is also not possible to decompose $Q^-_{n,i}$ (resp. $Q^+_{n,i}$) into union of intervals $\{Q^-\}$ (resp. $\{Q^+\}$) (of length multiple of n^2) such that

$$\max\Big(\sum_{Q^+ \in Q^+_{n,i}} s(Q^+)^\rho, \sum_{Q^- \in Q^-_{n,i}} s(Q^-)^\rho \Big) < s(Q_{n,i})^\rho/2.$$

In other words, one necessarily has

$$\max\big(s_\rho(Q^+_{n,i}), s_\rho(Q^-_{n,i}) \big) \geq s(Q_{n,i})^\rho/2.$$

One deduces by summation over i that

$$\max\big(s_\rho(Q^+_n), s_\rho(Q^-_n) \big) \geq \frac{1}{2} \sum_{i=1}^{m_n} \left(\frac{s(Q_{n,i})}{2^n} \right)^\rho. \tag{25}$$

Changing a little bit the point of view, considering any set Q_n^+ of intervals covering the set $\left(\bigcup_{x\in[0,\log\log n/2]}\mathcal{Z}_x\right)\cap\mathcal{S}_n$ and any set Q_n^- of intervals covering $\left(\bigcup_{x\in[-\log\log n/2,0]}\mathcal{Z}_x\right)\cap\mathcal{S}_n$, and using the same idea as the one which led to (25), the inequality (25) holds necessarily.

Finally, summing over n, one has

$$\max\left(\sum_{n\geq 1} s_\rho(Q_n^+), \sum_{n\geq 1} s_\rho(Q_n^-)\right) \geq \frac{1}{2}\sum_{n\geq 1}\sum_{i=1}^{m_n} s_\rho(Q_n) = +\infty,$$

this being true for any coverings Q_n^+ and Q_n^- defined as above.

One concludes that at least one of the two sums $\sum_{n=1}^{+\infty} s_\rho(Q_n^+)$ or $\sum_{n=1}^{+\infty} s_\rho(Q_n^-)$ is $+\infty$.

Assume that this is the first one: $\displaystyle\sum_{n\geq 1} s_\rho(Q_n^-) = +\infty$.

Let $x<0$ and consider $\mathcal{Z}_x = \{t\geq 0 : B_t = x\}$.

Assume also that N (given by Lemma 9) is so large that $(\log\log N)/2 > x$.

Let R_n be a covering of $\mathcal{Z}_x\cap\mathcal{S}_n$ by intervals $R_{n,i}$ of length that are multiple of n^2 (the sets Q_n and $Q_{n,i}$ are used for the optimal covering of $\mathcal{Z}$).

We use Remark 2. By the construction above, necessarily each union of intervals $Q_{n,i}^-$ is contained in $\bigcup_{R_{n,i}\in R_n}(R_{n,i}\pm 2n^2)$ (recall that the notation $R_{n,i}\pm n^2$ means that the interval $R_{n,i}$ enlarged by an interval of size $2n^2$ on both of its sides). Indeed, using the notations of (21), when $Q\subset R_{n,i}$ is an interval of size 1 such that the image of $\widetilde{Q} = Q\pm n^2$ by B contains $[-\log\log n/2, 0]$, necessarily $\mathcal{Z}_x\cap\widetilde{Q}\neq\emptyset$. We deduce that $Q_{n,i}^-\subset\bigcup_{R_{n,i}\in R_n}(R_{n,i}\pm n^2)$.

Denote by $\widetilde{R}_n = \bigcup_{R_{n,i}\in R_n} R_{n,i}\pm n^2$.

Remark 3. *Observe that by changing $R_{n,i}$ into $R_{n,i}\pm n^2$, $s_\rho(\widetilde{R}_n)$ is simply $s_\rho(R_n)$ multiplied by a factor $\in[1,5^\rho]$, which will not change the nature (convergence/divergence) of the series involved in the computation of the macroscopic dimension.*

The last lines of computations prove that

$$s_\rho(\widetilde{R}_n)\geq 5^{-\rho}s_\rho(Q_n^-),$$

which holds for any covering R_n of $\mathcal{Z}_x\cap\mathcal{S}_n$.

We conclude that for any sequence $(R_n)_{n\geq 1}$, with R_n a covering of $\mathcal{Z}_x\cap\mathcal{S}_n$ by intervals with length multiple of n^2 (and larger than $5n^2$), one necessarily has

$$\sum_{n\geq 1} s_\rho(\widetilde{R}_n)\geq 5^{-\rho}\sum_{n\geq 1} s_\rho(Q_n^-)\geq\frac{1}{2\cdot 5^\rho}\sum_{n\geq 1} s_\rho(Q_n) = +\infty.$$

Hence, $\mathrm{Dim}_H(\mathcal{Z}_x)\geq\rho$. One also sees that the argument holds simultaneously for all $x\leq 0$, almost surely.

Finally, applying the previous result to a sequence $(\rho_n)_{n\geq 1}$ which tends to $1/2$ leads to $\mathrm{Dim}_H(\mathcal{Z}_x)\geq 1/2$, for all $x\leq 0$. □

Now we prove Theorem 2.

Proof. The upper bound follows from the fact that for every $x\in\mathbb{R}$, $\mathcal{Z}_x$ is ultimately included in $E(\gamma)$ for every $\gamma>0$, and Theorem 1 (formula (2)) yields $\mathrm{Dim}_H(\mathcal{Z}_x)\leq 1/2$.

We turn to the lower bound. Let $(x_n)_{n\geq 1}$ be a dense sequence of real numbers. With probability one, the results of Lemma 10 apply to all the x_n's simultaneously. Define

$$Y = \sup\{y \in \mathbb{R} : \forall\, x_n < y,\ \mathrm{Dim}_{\mathrm{H}}(\mathcal{Z}_x) \geq 1/2 \text{ for all } x \leq x_n\}$$

with the convention that the supremum of an empty set is $-\infty$. One sees that, almost surely, there are only three possibilities:

(i) $Y = +\infty$, necessarily, for every $x \in \mathbb{R}$, $\mathrm{Dim}_{\mathrm{H}}(\mathcal{Z}_x) \geq 1/2$;

(ii) $Y = -\infty$, in this case, for each $y \in \mathbb{R}$, there exists $x_n \geq y$ such that $\mathrm{Dim}_{\mathrm{H}}(\mathcal{Z}_x) \geq 1/2$ for all $x \leq x_n$. So, for every $x \in \mathbb{R}$, $\mathrm{Dim}_{\mathrm{H}}(\mathcal{Z}_x) \geq 1/2$;

(iii) $-\infty < Y < +\infty$, consequently, for each $x_n < Y$, $\mathrm{Dim}_{\mathrm{H}}(\mathcal{Z}_x) \geq 1/2$ for all $x \leq x_n$, and for each $x_n > Y$, $\mathrm{Dim}_{\mathrm{H}}(\mathcal{Z}_x) \geq 1/2$ for all $x \geq x_n$. This proves that for every $x \in \mathbb{R} \setminus \{Y\}$, $\mathrm{Dim}_{\mathrm{H}}(\mathcal{Z}_x) \geq 1/2$.

□

6. Concluding remarks

In this article, we considered the large scale structure of certain sojourn sets where a Brownian motion visits "exceptionally" small values in a large scale dimension sense. Along the way, we use quite specific properties of the Brownian motion. For instance, the explicit probability density function of Brownian motion and $\frac{1}{2}$-stable subordinator are used in Lemma 3 and 6. It would be very interesting to extend the results here to general Lévy processes, where the densities (when they exist) are not explicitly known.

We believe that Theorem 2 holds for all level sets simultaneously. The problem is a lack of symmetry in the current proof. It would be interesting to find an alternative method to remove the unique point of exception (if any) in our result.

Acknowledgments

The first-named author was partly supported by the grant ANR MUTADIS ANR-11-JS01-0009. The second-named author was supported by DIM grants from Région Ile-de-France. The authors wish to thank Yimin Xiao for his insightful remarks on the subject.

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Stéphane Seuret, Université Paris-Est, LAMA(UMR8050), UPEMLV, UPEC, CNRS, F-94010 Créteil, France.
E-mail address: seuret@u-pec.fr

Xiaochuan Yang, Université Paris-Est, LAMA(UMR8050), UPEMLV, UPEC, CNRS, F-94010 Créteil, France. Present address: Dept. Statistics & Probability, Michigan State University, 48824 East Lansing, MI, USA
E-mail address: xiaochuan.j.yang@gmail.com