MULTIFRACTAL PROPERTIES OF SAMPLE PATHS OF GROUND STATE-TRANSFORMED JUMP PROCESSES

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Abstract. We consider a class of non-local Schrödinger operators and, using the ground state of such an operator, we define a random process generated by a unitary equivalent Lévy-type operator with unbounded coefficients. We construct this càdlàg process and show that it satisfies a related stochastic differential equation with jumps. Making use of this SDE we derive and prove the multifractal spectrum of local Hölder exponents of sample paths of the process.

1. **Introduction**

Non-local evolution equations of the type

$$\begin{aligned} &\partial_t u(x,t) = (-L + c(x,t))u(x,t), \quad (x,t) \in \mathbb{R}^d \times [0,\infty), \\ &u(x,0) = F(x), \end{aligned} \tag{1.1}$$

with suitable operators L and functions $c(x,t)$ receive increasing attention in analysis, probability, and related fields, and attract wide-ranging applications in fundamental models of mathematical physics and elsewhere, involving features of the solutions not encountered in the realm of partial differential equations. In what follows, we will consider operators of the form

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 f}{\partial x_j \partial x_i}(x) + \lim_{\varepsilon \searrow 0}\int_{|z-x|>\varepsilon} (f(z)-f(x))\nu(z-x)\mathrm{d}z, \quad x \in \mathbb{R}^d, \tag{1.2}$$

where $A = (a_{ij})_{1\leq i,j\leq d}$ is a symmetric non-negative definite matrix, and $\nu(\mathrm{d}z) = \nu(z)\mathrm{d}z$ is a symmetric Radon measure, called Lévy measure, on $\mathbb{R}^d \setminus \{0\}$. (For details see Section 2 below.) A much researched specific case is the fractional Laplacian $L = (-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, which is used in modelling, for instance, anomalous diffusion. In this case, $c(x,t)$ is a dissipation term resulting from interactions of the moving particle with a non-homogenous medium. When $L = (-\Delta + m^2)^{1/2} - m$, equation (1.1) is the imaginary-time (semi-)relativistic Schrödinger equation with rest mass $m \geq 0$, under potential $c(x,t) = V(x)$. For the special choice $L = \frac{1}{2}\Delta$ we get the classical (local) diffusion and related Schrödinger equations, in which case (1.1) reduces to a PDE. There are further applications for other choices of L.

For each specific choice of the matrix A and of the measure ν, the operator L has the special property of being the infinitesimal generator of a Lévy process. In this paper we are interested in some fine details on the random processes derived from a probabilistic representation of the solutions of (1.1), that is, in which also a term dependent on $c(x,t)$ appears. For simplicity, throughout this paper we choose $c(x,t) = V(x)$ and call it a potential.

Key-words: jump processes, sample path properties, Hausdorff dimension, Feynman-Kac semigroups, non-local Schrödinger operators, ground states
2010 *MS Classification*: Primary 60J75, 60G17, 28A78; Secondary 47D08, 47G20

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The evolution semigroup related to the above equation can be studied by suitable Feynman-Kac type representations. For a large class of potentials (see details below) an expression of the solutions

$$u(x,t) = \mathbb{E}^x[e^{-\int_0^t V(X_s)ds}F(X_t)] =: T_tF(x), \quad x \in \mathbb{R}^d,\ t \geq 0,$$

holds [12, 32], where $(X_t)_{t\geq 0}$ is the $\mathbb{R}^d$-valued Lévy process generated by L, and the expectation is taken with respect to the path measure of the specific process. The related Feynman-Kac semigroup $\{T_t : t \geq 0\}$ has many convenient properties, allowing a far reaching study of the solutions or spectral properties of $H = -L + V$. However, it is not conservative in the sense that $T_t \mathbf{1}_{\mathbb{R}^d} \neq \mathbf{1}_{\mathbb{R}^d}$, $t > 0$, therefore the Lévy process "perturbed" by the function V is in general no longer a random process. Nevertheless, by a suitable Doob h-transform one can change the measure under which the perturbation by V does become a Markov process.

Suppose that H has a non-empty discrete component in its spectrum, and let φ_0 be its unique eigenfunction (called *ground state*) corresponding to the lowest-lying eigenvalue, i.e., $H\varphi_0 = \lambda_0\varphi_0$ with $\varphi_0 \in \operatorname{Dom} H$ and $\lambda_0 = \inf \operatorname{Spec} H$. Then the map $f \mapsto \varphi_0 f$, $f \in \operatorname{Dom} H$, defines a unitary transform from $L^2(\mathbb{R}^d, \varphi_0^2 dx)$ to $L^2(\mathbb{R}^d, dx)$. It can be shown, see Section 2 below, that the image $\widetilde{H}$ of $H - \lambda_0$ under this unitary map gives the negative of the infinitesimal generator $\widetilde{L}$ of a Markov process, and for suitable test functions we have

$$\begin{aligned}(\widetilde{L}f)(x) = \frac{1}{2}\sigma\nabla \cdot \sigma\nabla f(x) + \sigma\nabla \ln \varphi_0(x) \cdot \sigma\nabla f(x) + \int_{0<|z|\leq 1} \frac{\varphi_0(x+z) - \varphi_0(x)}{\varphi_0(x)} z \cdot \nabla f(x)\nu(z)\mathrm{d}z \\ + \int_{\mathbb{R}^d\setminus\{0\}} \big(f(x+z) - f(x) - z\cdot\nabla f(x)\mathbf{1}_{\{|z|\leq 1\}}\big)\frac{\varphi_0(x+z)}{\varphi_0(x)}\nu(z)\mathrm{d}z, \quad (1.3)\end{aligned}$$

where ν is the Lévy intensity and $A = \sigma\sigma^T$ is the diffusion matrix of the Lévy process generated by L, and where we use the notation $\sigma\nabla \cdot \sigma\nabla f(x) = \sum_{i,j=1}^d (\sigma\sigma^T)_{ij}\partial_{x_i}\partial_{x_j} f(x)$. We call the resulting process a *ground state-transformed process* (occasionally also called $P(\phi)_1$*-process* following the terminology of B. Simon [40]).

The ground state-transformed process is of Lévy-type (by which we do not necessarily mean Feller), as the effect of V gives rise to position-dependent drift and jump components, having almost surely càdlàg paths. However, in contrast with many cases of Lévy-type processes studied in the literature, the coefficients of $\widetilde{L}$ are generally unbounded. It is known that pseudo-differential operators G defined like

$$(Gf)(x) = -\int_{\mathbb{R}^d} e^{ix\cdot y} g(x,y)\widehat{f}(y)\mathrm{d}y, \quad f \in C_c^\infty(\mathbb{R}^d),$$

where the hat means Fourier transform, give rise to Feller processes under suitable conditions on the symbol $g(x,y)$. Whenever $C_c^\infty(\mathbb{R}^d) \subset \operatorname{Dom} G$ and G does generate a Feller process, the Courrège representation

$$g(x,y) = g(x,0) - ib(x)\cdot y + \frac{1}{2} y \cdot A(x) y + \int_{\mathbb{R}^d\setminus\{0\}} \big(1 - e^{iz\cdot y} + iz\cdot y\mathbf{1}_{\{|z|\leq 1\}}\big)\,\nu(x,\mathrm{d}z)$$

holds, where the coefficients $b(x)$, $A(x)$ and $\nu(x,\cdot)$ play the same role of drift vector, diffusion matrix and jump measure as for Lévy processes, with the essential difference that they are now position dependent [9, 16, 8]. Furthermore, whenever the condition

$$\sup_{x\in\mathbb{R}^d} |g(x,y)| \leq C(1+|y|^2), \quad y \in \mathbb{R}^d, \tag{1.4}$$

holds, with a constant $C > 0$, the symbol can be used to analyze various properties of the process generated by G [37, 8]. It is also known, however, that (1.4) implies that all of the coefficients

$b(x)$, $A(x)$, $\nu(x,\cdot)$ are bounded [36]. Recently, there is an increasing interest in working also with unbounded coefficients, see [14, 30, 7, 38, 31] and [8, Sect. 3.6]. The results below on ground state-transformed processes complement these efforts since our approach is not through an analysis of the symbol, and apart from a direct interest in this context, our class of processes has an immediate relevance in the study of spectral properties of related self-adjoint operators and model Hamiltonians as a bonus [13, 32].

In the present paper our goal is to study sample path regularity properties of ground state-transformed processes obtained for a large class of operators H. The typical long-time behaviour of such processes has been established in [25], which is useful also in characterizing the support of the related Gibbs path measures defined by the right hand side of the Feynman-Kac formula (for perturbations of symmetric α-stable processes see also [21]). While asymptotic behaviour on the long run is driven by the large jumps, regularity at short range depends on the ultraviolet properties of H involving the small jumps. It is reasonable to expect that at least under sufficiently "nice" perturbations V of $-L$, the regularity of paths of a ground state-transformed process inherits the regularity of the underlying Lévy process. However, since the drift generated by the perturbation may become rough, the challenge is to establish conditions on V under which path regularity remains stable. Results in [2, 43, 44], where $L = (-\Delta)^{s(x)}$ and $V \equiv 0$, i.e., stable-like processes generated by fractional Laplacians of variable order are considered, show that local behaviour may become very complex and instead of an almost sure rule it can be even dependent on the individual path.

To describe local path regularity, we study the multifractal spectrum of local Hölder exponents attained by the paths. Recall that given a locally bounded function $f : \mathbb{R} \to \mathbb{R}^d$, it is said to belong to the pointwise Hölder space $C^h(x_0)$ for $h > 0$ and $x_0 \in \mathbb{R}$ whenever there exist constants $c, \delta > 0$, and a polynomial P of deg$P < \lfloor h \rfloor$ such that for $x \in B(x_0, \delta)$,

$$|f(x) - P(x - x_0)| \leq c|x - x_0|^h.$$

Note that $f \in C^1(x_0)$ does not mean that f is continuously differentiable at x_0. The Hölder exponent of f at point x_0 is then defined by

$$H_f(x_0) = \sup\{h > 0 : f \in C^h(x_0)\}.$$

Consider the set

$$E_f(h) = \{x \in \mathbb{R} : H_f(x) = h\}.$$

The *multifractal spectrum* of f is the map

$$D_f : h \mapsto \dim_{\mathsf{H}} E_f(h),$$

where $\dim_{\mathsf{H}}$ denotes Hausdorff dimension, with the convention that $\dim_{\mathsf{H}} \emptyset = -\infty$.

The multifractal spectrum of random processes has been studied by various authors. The above objects can now be defined pathwise. For Brownian motion, the Hölder exponent equals $\frac{1}{2}$ everywhere [35], thus

$$D_B(h) = \begin{cases} 1 & \text{if } h = \frac{1}{2} \\ -\infty & \text{otherwise} \end{cases}$$

almost surely, in which case the multifractal reduces to a mono-fractal behaviour. For a general Lévy process $(X_t)_{t\geq 0}$,

$$X_t = bt + \sigma B_t + \int_0^t \int_{|z|\leq 1} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{|z|>1} zN(\mathrm{d}s, \mathrm{d}z), \tag{1.5}$$

where $b \in \mathbb{R}^d$ is the drift term, $\sigma\sigma^T$ is the $d \times d$ diffusion matrix, N is a Poisson measure, and $\widetilde{N}$ is the compensated Poisson measure in $\mathbb{R}^d$ with intensity given by the Lévy measure $\nu(\mathrm{d}z)$, the behaviour relates with the upper Blumenthal-Getoor index [6] given by

$$\beta_\nu = \inf\Big\{\gamma \geq 0 : \int_{|z|\leq 1} |z|^\gamma \nu(\mathrm{d}z) < \infty\Big\}, \tag{1.6}$$

describing the growth rate of the Lévy measure around zero. The integrability condition of Lévy measures implies that $\beta_\nu \in [0,2]$. Jaffard [19] has proved the following for a Lévy process with $\beta_\nu \in (0,2)$. If $\sigma \neq 0$, then

$$D_X^1(h) = \begin{cases} \beta_\nu h & \text{if } h < 1/2 \\ 1 & \text{if } h = 1/2 \\ -\infty & \text{otherwise} \end{cases} \tag{1.7}$$

almost surely, and if $\sigma = 0$, then

$$D_X^2(h) = \begin{cases} \beta_\nu h & \text{if } h \leq 1/\beta_\nu \\ -\infty & \text{otherwise} \end{cases} \tag{1.8}$$

almost surely. Balança [1] has shown that the same result holds also for $\beta_\nu = 2$. (An example of a one-dimensional pure jump Lévy process with $\beta_\nu = 2$ is one with intensity $\nu(z) = 1/(z^3|\log z|^a)$, $a > 1$.) Extensions to Lévy fields and time-changed Lévy processes can be found in [10, 3].

We note that there are many further fractal properties of jump processes addressed in the literature. We refer to [27, 34, 28, 26] and the references therein, and for a review see [41].

The remainder of this paper is organized as follows. In Section 2 we define the classes of Lévy processes and potentials we consider, and prove the existence and càdlàg property of related ground state-transformed processes. Then in Section 3 we show that these processes satisfy a stochastic differential equation with jumps, which we use to derive our results on the multifractal behaviour of the Hölder exponents of their paths.

2. Ground state-transformed jump processes

2.1. Lévy processes and perturbations by potentials

Let $(X_t)_{t\geq 0}$ be a rotationally symmetric Lévy process with values in $\mathbb{R}^d$, $d \geq 1$, i.e., as given by (1.5) in which we set $b = 0$. The probability measure of the process starting at $x \in \mathbb{R}^d$ will be denoted by $\mathbb{P}^x$, and expectation with respect to this measure by $\mathbb{E}^x$. The process $(X_t)_{t\geq 0}$ is determined by its characteristic function

$$\mathbb{E}^0\left[e^{iy\cdot X_t}\right] = e^{-t\psi(y)}, \quad y \in \mathbb{R}^d,\ t > 0,$$

with the characteristic exponent given by the Lévy-Khintchin formula

$$\psi(y) = \frac{1}{2}Ay\cdot y + \int_{\mathbb{R}^d}(1-\cos(y\cdot z))\nu(\mathrm{d}z). \tag{2.1}$$

Here $A = (a_{ij})_{1\leq i,j\leq d} = \sigma\sigma^T$ is a symmetric non-negative definite matrix, and ν is a symmetric Lévy measure on $\mathbb{R}^d\setminus\{0\}$, i.e., $\int_{\mathbb{R}^d}(1\wedge|z|^2)\nu(\mathrm{d}z) < \infty$ and $\nu(E) = \nu(-E)$, for every Borel set $E \subset \mathbb{R}^d\setminus\{0\}$, thus the Lévy triplet of the process is $(0, \frac{1}{2}A, \nu)$. We will assume throughout that the Lévy measure in (2.1) has infinite mass and it is absolutely continuous with respect to Lebesgue measure, i.e., $\nu(\mathbb{R}^d\setminus\{0\}) = \infty$ and $\nu(\mathrm{d}x) = \nu(x)\mathrm{d}x$, with density $\nu(x) > 0$.

The generator L of the process $(X_t)_{t\geq 0}$ is determined by its symbol ψ through

$$\widehat{Lf}(y) = -\psi(y)\widehat{f}(y), \quad y \in \mathbb{R}^d,\ f \in \mathrm{Dom}(L), \tag{2.2}$$

with domain $\mathrm{Dom}(L) = \big\{f \in L^2(\mathbb{R}^d) : \psi\widehat{f} \in L^2(\mathbb{R}^d)\big\}$. It is a negative, non-local, self-adjoint operator with core $C_{\rm c}^\infty(\mathbb{R}^d)$, and it has the expression (1.2) for $f \in C_{\rm c}^\infty(\mathbb{R}^d)$.

Next consider the set of functions

$$\mathcal{K}^X = \Big\{ f : \mathbb{R} \to \mathbb{R}^d : f \text{ is Borel measurable and } \lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \mathbb{E}^x\Big[\int_0^t |f(X_s)|ds\Big] = 0 \Big\}. \tag{2.3}$$

We say that the potential $V : \mathbb{R}^d \to \mathbb{R}$ belongs to *X-Kato class*, i.e., associated with the Lévy process $(X_t)_{t\geq 0}$, whenever it satisfies

$$V_- \in \mathcal{K}^X \quad \text{and} \quad V_+ \in \mathcal{K}^X_{\rm loc}, \quad \text{with} \quad V_+ = \max\{V, 0\},\ V_- = \min\{V, 0\},$$

where $V_+ \in \mathcal{K}^X_{\rm loc}$ means that $V_+ 1_B \in \mathcal{K}^X$ for all compact sets $B \subset \mathbb{R}^d$. It is straightforward to see that $L^\infty_{\rm loc}(\mathbb{R}^d) \subset \mathcal{K}^X_{\rm loc}$, moreover, by stochastic continuity of $(X_t)_{t\geq 0}$ also $\mathcal{K}^X_{\rm loc} \subset L^1_{\rm loc}(\mathbb{R}^d)$. Note that X-Kato class potentials may have local singularities.

By standard arguments based on Khasminskii's Lemma, see [32, Lem.3.37-3.38], for an X-Kato class potential V it follows that there exist suitable constants $C_1(X, V), C_2(X, V) > 0$ such that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^x\left[e^{-\int_0^t V(X_s){\rm d}s}\right] \leq \sup_{x\in\mathbb{R}^d} \mathbb{E}^x\left[e^{\int_0^t V_-(X_s){\rm d}s}\right] \leq C_1 e^{C_2 t}, \quad t > 0. \tag{2.4}$$

This implies that

$$T_t f(x) = \mathbb{E}^x\left[e^{-\int_0^t V(X_s){\rm d}s} f(X_t)\right], \quad f \in L^2(\mathbb{R}^d),\ t > 0,$$

are well defined operators. Using the Markov property and stochastic continuity of $(X_t)_{t\geq 0}$ it can be shown that $\{T_t : t \geq 0\}$ is a strongly continuous semigroup of symmetric operators on $L^2(\mathbb{R}^d)$, which we call the *Feynman-Kac semigroup* associated with the process $(X_t)_{t\geq 0}$ and potential V. In particular, by the Hille-Yoshida theorem there exists a self-adjoint operator H, bounded from below, such that $e^{-tH} = T_t$, with core $C_{\rm c}^\infty(\mathbb{R}^d)$. We call the operator H a *non-local Schrödinger operator* whose kinetic term is the negative of the infinitesimal generator L of the process $(X_t)_{t\geq 0}$. Since any X-Kato class potential is relatively form bounded with respect to $-L$ with relative bound less than 1, we have

$$H = -L \dot{+} V,$$

where the dot means form sense, and V acts as a multiplication operator [32, Ch. 3]. (Formally, we will write just an ordinary sum having this meaning of the operator.) For instance, when $(X_t)_{t\geq 0}$ is a subordinate Brownian motion, we have $L = \Phi(-\Delta)$, where Φ is the Laplace exponent of the corresponding subordinator [12].

We make the following standing assumption throughout this paper (see remark below).

Assumption 2.1. *The operators L and V are chosen in such a way that the following conditions hold:*

1. *The self-adjoint operator $H = -L + V$ has a* ground state*, i.e., an eigenfunction $\varphi_0 \in \mathrm{Dom}\, H \subset L^2(\mathbb{R}^d)$ such that*

$$H\varphi_0 = \lambda_0\varphi_0, \quad \varphi_0 \not\equiv 0, \quad \lambda_0 = \inf \mathrm{Spec}\, H, \tag{2.5}$$

   and is normalized by $\|\varphi_0\|_2 = 1$.

2. *The ground state φ_0 is unique, strictly positive, bounded and continuous, with a pointwise decay to zero at infinity.*

Remark 2.1. From (2.1)-(2.2) we have that $\mathrm{Spec}(-L) = \mathrm{Spec}_{\mathrm{ess}}(-L) = [0,\infty)$. It follows by general arguments that whenever the potential is confining, i.e., $V(x) \to \infty$ as $|x| \to \infty$, the spectrum of $-L$ completely changes under the perturbation and the spectrum of H becomes purely discrete, consisting of isolated eigenvalues of finite multiplicities. Thus for confining potentials a ground state (2.5) always exists and part (1) of Assumption 2.1 holds. When the potential is decaying, i.e., $V(x) \to 0$ as $|x| \to \infty$, or it is confining in one direction and decaying in another direction, the spectrum of H may or may not contain a discrete component, and the existence of a ground state depends on further details of V. Again, by general results it follows that whenever a ground state of H does exist, part (2) of Assumption 2.1 holds. For details we refer to [32, Ch. 3] and [22, 24].

2.2. **Existence and càdlàg property of ground state-transformed processes**

By using $\varphi_0 > 0$, we define the *ground state transform* as the unitary map

$$U : L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x) \to L^2(\mathbb{R}^d, \mathrm{d}x), \quad f \mapsto \varphi_0 f.$$

Also, we define the intrinsic Feynman-Kac semigroup

$$\widetilde{T}_t f(x) = \frac{e^{\lambda_0 t}}{\varphi_0(x)} T_t(\varphi_0 f)(x) \tag{2.6}$$

associated with $\{T_t : t \geq 0\}$. Using the integral kernel $u(t,x,y)$ of T_t we then have that $\widetilde{T}_t f(x) = \int_{\mathbb{R}^d} \tilde{u}(t,x,y) f(y) \varphi_0^2(y) \mathrm{d}y$ with the integral kernel given by

$$\widetilde{u}(t,x,y) = \frac{e^{\lambda_0 t} u(t,x,y)}{\varphi_0(x)\varphi_0(y)}, \tag{2.7}$$

and infinitesimal generator $\widetilde{L} = -\widetilde{H}$, where

$$\widetilde{H} = U^{-1}(H - \lambda_0)U, \tag{2.8}$$

with domain

$$\mathrm{Dom}\, \widetilde{H} = \big\{ f \in L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x) : Uf \in \mathrm{Dom}\, H \big\}.$$

A calculation gives the expression (1.3), which holds at least for C_{c}^∞ functions. The operators $\widetilde{T}_t = e^{t\widetilde{L}}$ are contractions and we have $\widetilde{T}_t 1_{\mathbb{R}^d} = 1_{\mathbb{R}^d}$ for all $t \geq 0$, thus $\{\widetilde{T}_t : t \geq 0\}$ is a Markov semigroup on $L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x)$.

The self-adjoint operator $\widetilde{L}$ generates a stationary strong Markov process, which we call *ground state-transformed (GST) process* and denote by $(\widetilde{X}_t)_{t\geq 0}$. GST processes have been constructed first for Brownian motion perturbed by potentials, see [40, 5] and [32, Sects. 4.10.2, 4.11.9] for further details and applications. However, due to the jumps in our case there are some essential modifications, and we give a proof of the existence of a càdlàg version of GST jump processes in the generality allowed by Assumption 2.1.

Denote by Ω_{r} the space of right continuous functions from $[0,\infty)$ to $\mathbb{R}^d$ with left limits (i.e., càdlàg functions), and by Ω_{l} the space of left continuous functions from $[0,\infty)$ to $\mathbb{R}^d$ with right limits (i.e., càglàd functions). Denote the corresponding Borel σ-fields by $\mathcal{B}(\Omega_{\mathrm{r}})$ and $\mathcal{B}(\Omega_{\mathrm{l}})$, respectively. Also, denote by Ω the space of càdlàg functions from $\mathbb{R}$ to $\mathbb{R}$, and its Borel σ-field by $\mathcal{B}(\Omega)$.

Theorem 2.1. *Let $(X_t)_{t\geq 0}$ be a Lévy process with generator L as given by (2.1)-(2.2), V be an X-Kato class potential, and suppose that $H = -L + V$ has a ground state φ_0. For all $x \in \mathbb{R}^d$ there exists a probability measure $\widetilde{\mathbb{P}}^x$ on $(\Omega, \mathcal{B}(\Omega))$ and a random process $(\widetilde{X}_t)_{t\in\mathbb{R}}$ satisfying the following properties:*

(1) *Let $-\infty < t_0 \leq t_1 \leq ... \leq t_n < \infty$ be an arbitrary division of the real line, for any $n \in \mathbb{N}$. The initial distribution of the process is*

$$\widetilde{\mathbb{P}}^x(\widetilde{X}_0 = x) = 1,$$

and the finite dimensional distributions of $\widetilde{\mathbb{P}}^x$ with respect to the stationary distribution $\varphi_0^2 \mathrm{d}x$ are given by

$$\int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=0}^n f_j(\widetilde{X}_{t_j})\Big]\varphi_0^2(x)\mathrm{d}x = \Big(f_0,\ \widetilde{T}_{t_1-t_0}\, f_1 ... \widetilde{T}_{t_n-t_{n-1}}\, f_n\Big)_{L^2(\mathbb{R}^d,\varphi_0^2\mathrm{d}x)} \tag{2.9}$$

for all $f_0, f_n \in L^2(\mathbb{R}^d, \varphi_0^2\mathrm{d}x)$, $f_j \in L^\infty(\mathbb{R}^d)$, $j = 1, ..., n-1$.

(2) *The finite dimensional distributions are time-shift invariant, i.e.,*

$$\int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=0}^n f_j(\widetilde{X}_{t_j})\Big]\varphi_0^2(x)\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=0}^n f_j(\widetilde{X}_{t_j+s})\Big]\varphi_0^2(x)\mathrm{d}x, \quad s \in \mathbb{R}, n \in \mathbb{N}.$$

(3) *$(\widetilde{X}_t)_{t\geq 0}$ and $(\widetilde{X}_t)_{t\leq 0}$ are independent, and $\widetilde{X}_{-t} \overset{\mathrm{d}}{=} \widetilde{X}_t$, for all $t \in \mathbb{R}$.*

(4) *Consider the filtrations $\left(\mathcal{F}_t^+\right)_{t\geq 0} = \sigma\left(\widetilde{X}_s : 0 \leq s \leq t\right)$ and $\left(\mathcal{F}_t^-\right)_{t\leq 0} = \sigma\left(\widetilde{X}_s : t \leq s \leq 0\right)$. Then $(\widetilde{X}_t)_{t\geq 0}$ is a Markov process with respect to $\left(\mathcal{F}_t^+\right)_{t\geq 0}$, and $(\widetilde{X}_t)_{t\leq 0}$ is a Markov process with respect to $\left(\mathcal{F}_t^-\right)_{t\leq 0}$.*

Furthermore, we have for all $f, g \in L^2(\mathbb{R}^d, \varphi_0^2\mathrm{d}x)$ the change-of-measure formula

$$(f, \widetilde{T}_t g)_{L^2(\mathbb{R}^d,\varphi_0^2\mathrm{d}x)} = (f\varphi_0, e^{-t(H-\lambda_0)}g\varphi_0)_{L^2(\mathbb{R}^d,\mathrm{d}x)} = \int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}[f(\widetilde{X}_0)g(\widetilde{X}_t)]\varphi_0^2(x)\mathrm{d}x, \quad t \geq 0. \tag{2.10}$$

The probability measure $\widetilde{\mathbb{P}}^x$ is a Gibbs measure on the space of two-sided càdlàg paths, see a discussion for stable processes in [21, Sect. 5.3]. For the remaining part of this section we present a proof of this theorem.

Let $n \in \mathbb{N}$ be arbitrary, and consider any time division $0 \leq t_0 \leq t_1 \leq ... \leq t_n$. Define the set function $P_{\{t_0,...,t_n\}} : \times_{j=0}^n \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}$ by

$$P_{\{t_0\}}(A_0) = \Big(\mathbf{1}, \widetilde{T}_{t_0}\mathbf{1}_{A_0}\Big)_{L^2(\mathbb{R}^d,\varphi_0^2\mathrm{d}x)} = (\mathbf{1}, \mathbf{1}_{A_0})_{L^2(\mathbb{R}^d,\varphi_0^2\mathrm{d}x)} \tag{2.11}$$

$$P_{\{t_0,...,t_n\}}(\times_{j=0}^n A_j) = \Big(\mathbf{1}_{A_0}, \widetilde{T}_{t_1-t_0}\mathbf{1}_{A_1}...\mathbf{1}_{A_{n-1}}\widetilde{T}_{t_n-t_{n-1}}\mathbf{1}_{A_n}\Big)_{L^2(\mathbb{R}^d,\varphi_0^2\mathrm{d}x)}, \quad n \in \mathbb{N}, \tag{2.12}$$

with arbitrary Borel sets $A_0, ..., A_n \in \mathcal{B}(\mathbb{R}^d)$.

Step 1: First we obtain a probability measure on the set of all functions $[0, \infty) \to \mathbb{R}^d$ by a projective limit of the prescribed finite dimensional distributions (2.11)-(2.12), which is a standard step. Denote the set of finite subsets of the positive semi-axis by $\mathcal{P}_f(\mathbb{R}^+) = \{\Lambda \subset [0, \infty) : |\Lambda| < \infty\}$, where the bars denote cardinality of the set. It can be verified directly that the family of set functions $(P_\Lambda)_{\Lambda\in\mathcal{P}_f(\mathbb{R}^+)}$ satisfies the consistency condition of the marginals

$$P_{\{t_0,...,t_{n+m}\}}\Big((\times_{j=0}^n A_j) \times (\times_{j=n+1}^{n+m}\mathbb{R}^d)\Big) = P_{\{t_0,...,t_n\}}(\times_{j=0}^n A_j), \quad n, m \in \mathbb{N}.$$

Hence by the Kolmogorov extension theorem there exists a probability measure P_∞ and a random process $(Z_t)_{t\geq 0}$ on the measurable space $\big((\mathbb{R}^d)^{[0,\infty)}, \sigma(\mathcal{A})\big)$, where

$$\mathcal{A} = \Big\{\omega : \mathbb{R} \to \mathbb{R}^d :\ \mathrm{Ran}\ \omega\big|_\Lambda \subset E,\ E \in (\mathcal{B}(\mathbb{R}^d))^{|\Lambda|},\ \Lambda \in \mathcal{P}_f(\mathbb{R}^+)\Big\}, \tag{2.13}$$

such that

$$P_{\{t_0\}}(A) = \mathbb{E}_{P_\infty}[\mathbf{1}_A(Z_{t_0})] \quad \text{and} \quad P_{\{t_0,\dots,t_n\}}\left(\times_{j=0}^n A_j\right) = \mathbb{E}_{P_\infty}\Big[\prod_{j=0}^n \mathbf{1}_{A_j}(Z_{t_j})\Big], \quad n \in \mathbb{N},$$

hold. Hence we have

$$\mathbb{E}_{P_\infty}[f_0(Z_{t_0})] = \left(1, \widetilde{T}_{t_0} f_0\right)_{L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x)} = (\mathbf{1}, f_0)_{L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x)} \tag{2.14}$$

$$\mathbb{E}_{P_\infty}\Big[\prod_{j=0}^n f_j(Z_{t_j})\Big] = \left(f_0, \widetilde{T}_{t_1-t_0} f_1 \dots \widetilde{T}_{t_n-t_{n-1}} f_n\right)_{L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x)}, \tag{2.15}$$

for $f_j \in L^\infty(\mathbb{R}^d)$, $j = 1, \dots, n-1$, $f_0, f_n \in L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x)$, $0 \le t_0 \le t_1 \le \dots \le t_n$, and all $n \in \mathbb{N}$.

Step 2: Next we prove the existence of both a càdlàg and a càglàd version of $(Z_t)_{t\ge 0}$. In this step we show that the Dynkin-Kinney condition holds.

Lemma 2.1. *Let $T > 0$ be arbitrary but fixed. Then for every $\varepsilon > 0$ we have $P_\infty(|Z_t - Z_s| > \varepsilon) \to 0$ as $|t-s| \to 0$, uniformly in $s, t \in [0, T]$.*

Proof. We write the right hand side of (2.15) in terms of $(X_t)_{t\ge0}$, i.e.,

$$\mathbb{E}_{P_\infty}\Big[\prod_{j=0}^n f_j(Z_{t_j})\Big] = \int_{\mathbb{R}^d} \mathbb{E}^x\Big[e^{-\int_0^{t_n}(V(X_s)-\lambda_0)\mathrm{d}s} \prod_{j=0}^n f_j(X_{t_j}))\varphi_0(X_{t_n})\Big]\varphi_0(x)\mathrm{d}x. \tag{2.16}$$

Recall that $\mathbb{E}^x$ is expectation with respect to the measure $\mathbb{P}^x$ of the Lévy process $(X_t)_{t\ge0}$. Let $0 \le s < t \le T$, and denote by $B_\varepsilon(x)$ the ball of radius ε centered in x. By (2.16), the Markov property of $(X_t)_{t\ge0}$, and the conservative property of the intrinsic semigroup $\{\widetilde{T}_t : t \ge 0\}$ on $L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x)$, we have

$$P_\infty(|Z_t - Z_s| > \varepsilon) = \int_{\mathbb{R}^d} \mathbb{E}^x\left[e^{-\int_0^{t-s}(V(X_r)-\lambda_0)\mathrm{d}r}\varphi_0(X_{t-s})\mathbf{1}_{B_\varepsilon(x)^c}(X_{t-s})\right]\varphi_0(x)\mathrm{d}x.$$

Schwarz inequality gives

$$\begin{aligned} &P_\infty(|Z_t - Z_s| > \varepsilon) \\ &\le \int_{\mathbb{R}^d} \left(\mathbb{E}^x\left[\varphi_0^2(X_{t-s})\right]\right)^{1/2} \left(\mathbb{E}^x\left[\mathbf{1}_{B_\varepsilon(x)^c}(X_{t-s}) e^{-2\int_0^{t-s}(V(X_r)-\lambda_0)\mathrm{d}r}\right]\right)^{1/2} \varphi_0(x)\mathrm{d}x. \end{aligned} \tag{2.17}$$

Using again Schwarz inequality, we have

$$\begin{aligned} &\mathbb{E}^x\left[\mathbf{1}_{B_\varepsilon(x)^c}(X_{t-s}) e^{-2\int_0^{t-s}(V(X_r)-\lambda_0)\mathrm{d}r}\right] \\ &\quad \le \left(\mathbb{E}^x\left[\mathbf{1}_{B_\varepsilon(x)^c}(X_{t-s})\right]\right)^{1/2} \left(\mathbb{E}^x\left[e^{-4\int_0^{t-s}(V(X_r)-\lambda_0)\mathrm{d}r}\right]\right)^{1/2} \le C\,\mathbb{P}(|X_{t-s}| > \varepsilon)^{1/2}, \end{aligned}$$

where $C = \sup_{x\in\mathbb{R}^d}\left(\mathbb{E}^x\left[e^{-4\int_0^{t-s}(V(X_r)-\lambda_0)\mathrm{d}r}\right]\right)^{1/2}$. Thus by (2.17) and a repeated use of Schwarz inequality we get

$$P_\infty(|Z_t - Z_s| > \varepsilon) \le C^{1/2}\mathbb{P}(|X_{t-s}| > \varepsilon)^{1/4}\|\varphi_0\|_2^2,$$

which goes to zero as $|t-s| \to 0$ by stochastic continuity of $(X_t)_{t\ge0}$.

To show uniform convergence, fix $\eta > 0$. By stochastic continuity, for every t there exists $\rho_t > 0$ such that $P_\infty(|Z_t - Z_s| \ge \frac{\varepsilon}{2}) \le \frac{\eta}{2}$ for $|s-t| < \rho_t$. Let $I_t = (t - \frac{\rho_t}{2}, t + \frac{\rho_t}{2})$. There is a finite covering I_{t_j}, $j = 1, \dots, n$, such that $\cup_{j=1}^n I_{t_j} \supset [0, T]$. Let $\rho = \min_{1\le j\le n}\rho_{t_j}$. If $|s-t| < \rho$ and $s, t \in [0, T]$, then $t \in I_{t_j}$ for some j, hence $|s - t_j| < \rho_{t_j}$ and

$$P_\infty(|Z_t - Z_s| > \varepsilon) \le P_\infty(|Z_t - Z_{t_j}| > \varepsilon/2) + P_\infty(|Z_s - Z_{t_j}| > \varepsilon/2) < \eta.$$

□

Recall that $\Omega_{\rm r}$ and $\Omega_{\rm l}$ denote the càdlàg and càglàd path spaces over $[0,\infty)$, respectively. By Lemma 2.1 there exists a càdlàg version $\bar{Z}=(\bar{Z}_t)_{t\geq 0}$ of $(Z_t)_{t\geq 0}$ on the space $((\mathbb{R}^d)^{[0,\infty)},\sigma(\mathcal{A}),P_\infty)$. Denote the image measure of P_∞ on $(\Omega_{\rm r},\mathcal{B}(\Omega_{\rm r}))$ by $Q_{\rm r}=P_\infty\circ\bar{Z}^{-1}$. Let $(Y_t)_{t\geq 0}$ be the coordinate process on $(\Omega_{\rm r},\mathcal{B}(\Omega_{\rm r}),Q_{\rm r})$ such that $\bar{Z}_t\stackrel{\rm d}{=}Y_t$. In terms of $(Y_t)_{t\geq 0}$, equalities (2.14)-(2.15) become

$$\Big(\mathbf{1},\widetilde{T}_{t_0}f_0\Big)_{L^2(\mathbb{R}^d,\varphi_0^2{\rm d}x)}=(\mathbf{1},f_0)_{L^2(\mathbb{R}^d,\varphi_0^2{\rm d}x)}=\mathbb{E}_{Q_{\rm r}}[f_0(Y_{t_0})], \tag{2.18}$$

$$\Big(f_0,\widetilde{T}_{t_1-t_0}f_1...f_{n-1}\widetilde{T}_{t_n-t_{n-1}}f_n\Big)_{L^2(\mathbb{R}^d,\varphi_0^2{\rm d}x)}=\mathbb{E}_{Q_{\rm r}}\Big[\prod_{j=0}^n f_j(Y_{t_n})\Big]. \tag{2.19}$$

Similarly, a càglàd version $\underline{Z}=(\underline{Z}_t)_{t\geq 0}$ of $(Z_t)_{t\geq 0}$ can be also constructed on the same probability space $((\mathbb{R}^d)^{[0,\infty)},\sigma(\mathcal{A}),P_\infty)$. Likewise, there exists a probability measure $Q_{\rm l}$ on the càglàd space $(\Omega_{\rm l},\mathcal{B}(\Omega_{\rm l}))$ such that the coordinate process $(Y_t)_{t\geq 0}$ satisfies (2.18)-(2.19) with $Q_{\rm l}$, and $Q_{\rm l}=P_\infty\circ\underline{Z}^{-1}$ holds.

Step 3: We define the regular conditional probability measures $Q_{\rm r}^x(\,\cdot\,)=Q_{\rm r}(\,\cdot\,|Y_0=x)$ and $Q_{\rm l}^x(\,\cdot\,)=Q_{\rm l}(\,\cdot\,|Y_0=x)$ for $x\in\mathbb{R}^d$ on $(\Omega_{\rm r},\mathcal{B}(\Omega_{\rm r}))$ and $(\Omega_{\rm l},\mathcal{B}(\Omega_{\rm l}))$, respectively. Since Y_0 is distributed by $\varphi_0^2(x){\rm d}x$, we have $Q_{\rm r}(A)=\int_{\mathbb{R}^d}\mathbb{E}_{Q_{\rm r}^x}[\mathbf{1}_A]\varphi_0^2(x){\rm d}x$ and $Q_{\rm l}(A)=\int_{\mathbb{R}^d}\mathbb{E}_{Q_{\rm l}^x}[\mathbf{1}_A]\varphi_0^2(x){\rm d}x$. Hence the process $(Y_t)_{t\geq 0}$ on $(\Omega_{\rm r},\mathcal{B}(\Omega_{\rm r}),Q_{\rm r}^x)$ satisfies

$$\Big(\mathbf{1},\widetilde{T}_{t_0}f_0\Big)_{L^2(\mathbb{R}^d,\varphi_0^2{\rm d}x)}=(\mathbf{1},f_0)_{L^2(\mathbb{R}^d,\varphi_0^2{\rm d}x)}=\int_{\mathbb{R}^d}\mathbb{E}_{Q_{\rm r}^x}[f_0(Y_{t_0})]\varphi_0^2(x){\rm d}x \tag{2.20}$$

$$\Big(f_0,\widetilde{T}_{t_1-t_0}f_1...f_{n-1}\widetilde{T}_{t_n-t_{n-1}}f_n\Big)_{L^2(\mathbb{R}^d,\varphi_0^2{\rm d}x)}=\int_{\mathbb{R}^d}\mathbb{E}_{Q_{\rm r}^x}\Big[\prod_{j=0}^n f_j(Y_{t_j})\Big]\varphi_0^2(x){\rm d}x, \tag{2.21}$$

and the process $(Y_t)_{t\geq 0}$ on $(\Omega_{\rm l},\mathcal{B}(\Omega_{\rm l}),Q_{\rm l}^x)$ satisfies (2.20)-(2.21) with $Q_{\rm l}^x$.

Lemma 2.2. *The coordinate process $(Y_t)_{t\geq 0}$ is a Markov process on $(\Omega_{\rm r},\mathcal{B}(\Omega_{\rm r}),Q_{\rm r}^x)$ with respect to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$. Similarly, the coordinate process $(Y_t)_{t\geq 0}$ is a Markov process on the probability space $(\Omega_{\rm l},\mathcal{B}(\Omega_{\rm l}),Q_{\rm l}^x)$ with respect to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$.*

Proof. Let $q_t(x,A)=\widetilde{T}_t\mathbf{1}_A(x)$, for every $A\in\mathcal{B}(\mathbb{R}^d)$, $x\in\mathbb{R}^d$ and $t\geq 0$. Clearly, $q_t(x,A)=\mathbb{E}_{Q_{\rm r}^x}[\mathbf{1}_A(Y_t)]$ and by (2.20)-(2.21) the finite dimensional distributions of $(Y_t)_{t\geq 0}$ can be written as

$$\mathbb{E}_{Q_{\rm r}^x}\Big[\prod_{j=0}^n\mathbf{1}_{A_j}(Y_{t_j})\Big]=\int_{\mathbb{R}^d}\prod_{j=0}^n\mathbf{1}_{A_j}(x_j)q_{t_j-t_{j-1}}(x_{j-1},{\rm d}x_j), \tag{2.22}$$

with $t_0=0$ and $x_0=x$. By using the properties of the semigroup $\{\widetilde{T}_t:t\geq 0\}$, it is checked directly that $q_t(x,A)$ is a probability transition kernel, thus $(Y_t)_{t\geq 0}$ is a Markov process with finite dimensional distributions given by (2.22). The second statement can be proven similarly. □

Step 4: Next we construct a random process indexed by the whole real line $\mathbb{R}$. Consider $\widehat{\Omega}=\Omega_{\rm r}\times\Omega_{\rm l}$ with product σ-field $\widehat{\mathcal{F}}=\mathcal{B}(\Omega_{\rm r})\times\mathcal{B}(\Omega_{\rm l})$ and product measure $\widehat{Q}^x=Q_{\rm r}^x\times Q_{\rm l}^x$. Define the coordinate process $(\widehat{Y}_t)_{t\geq 0}$ by

$$\widehat{Y}_t(\omega)=\begin{cases}\omega_1(t) & t\geq 0\\ \omega_2(-t) & t<0\end{cases}$$

for $\omega=(\omega_1,\omega_2)\in\widehat{\Omega}$. This is then a random process $(\widehat{Y}_t)_{t\in\mathbb{R}}$ on $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{Q}^x)$ such that $\widehat{Q}^x(\widehat{Y}_0=x)=1$ and $\mathbb{R}\ni t\mapsto\widehat{Y}_t(\omega)$ is càdlàg. It is direct to see that $\widehat{Y}_t$, $t\geq 0$, and $\widehat{Y}_s$, $s\leq 0$, are independent, and $\widehat{Y}_t\stackrel{\rm d}{=}\widehat{Y}_{-t}$.

Step 5: Denote the image measure of $\widehat{Q}^x$ on $(\Omega_{\rm r}, \mathcal{B}(\Omega_{\rm r}))$ with respect to $\widehat{Y} = (\widehat{Y}_t)_{t\in\mathbb{R}}$ by $\widetilde{\mathbb{P}}^x = \widehat{Q}^x \circ \widehat{Y}^{-1}$. Let $\widetilde{X}_t(\omega) = \omega(t)$, $t \in \mathbb{R}$, $\omega \in \Omega$, denote the coordinate process. Clearly, we have $\widetilde{X}_t \stackrel{\rm d}{=} Y_t$ for $t \in \mathbb{R}$. Thus we see that $\widetilde{X}_t \stackrel{\rm d}{=} \widetilde{X}_{-t}$, and by Step 4 above $(\widetilde{X}_t)_{t\geq 0}$ and $(\widetilde{X}_t)_{t\leq 0}$ are independent. Furthermore, by Step 2 we have that $(\widetilde{X}_t)_{t\geq 0}$ and $(\widetilde{X}_t)_{t\leq 0}$ are Markov processes with respect to $(\mathcal{F}_t^+)_{t\geq 0}$ and $(\mathcal{F}_t^-)_{t\leq 0}$, respectively.

To prove shift invariance, consider arbitrary time-points $t_0 \leq \ldots \leq t_n \leq 0 \leq t_{n+1} \leq \ldots \leq t_{n+m}$, $n, m \in \mathbb{N}$. Then by independence of $(\widetilde{X}_t)_{t\leq 0}$ and $(\widetilde{X}_t)_{t\geq 0}$ we have

$$\int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=0}^{n+m} f_j(\widetilde{X}_{t_j})\Big]\varphi_0^2 {\rm d}x = \int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=0}^{n} f_j(\widetilde{X}_{t_j})\Big]\mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=n+1}^{n+m} f_j(\widetilde{X}_{t_j})\Big]\varphi_0^2 {\rm d}x.$$

Moreover,

$$\mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=n+1}^{n+m} f_j(\widetilde{X}_{t_j})\Big] = (\widetilde{T}_{t_{n+1}} f_{n+1} \widetilde{T}_{t_{n+2}-t_{n+1}} f_{n+2} \ldots \widetilde{T}_{t_{n+m}-t_{n+m-1}} f_{n+m})(x)$$

and

$$\mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=0}^{n+m} f_j(\widetilde{X}_{t_j})\Big] = \mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=0}^{n+m} f_j(\widetilde{X}_{-t_j})\Big] = (\widetilde{T}_{t_n} f_n \widetilde{T}_{t_n-t_{n-1}} f_{n-1} \ldots \widetilde{T}_{t_1-t_0} f_1)(x).$$

A combination of the above gives

$$\begin{aligned}\int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=0}^{n+m} f_j(\widetilde{X}_{t_j})\Big]\varphi_0^2(x) {\rm d}x &= (\widetilde{T}_{t_n} f_n \ldots \widetilde{T}_{t_1-t_0} f_1, \widetilde{T}_{t_{n+1}} f_{n+1} \ldots \widetilde{T}_{t_{n+m}-t_{n+m-1}} f_{n+m})_{L^2(\mathbb{R}^d, \varphi_0^2 {\rm d}x)} \\ &= (f_1, \widetilde{T}_{t_1-t_0} f_2 \ldots \widetilde{T}_{t_{n+m}-t_{n+m-1}} f_{n+m})_{L^2(\mathbb{R}^d, \varphi_0^2 {\rm d}x)}.\end{aligned}$$

This implies the required time-shift invariance. Formula (2.10) is now a direct consequence, and this completes the proof of the theorem.

3. **Local path regularity of GST processes**

3.1. **Stochastic differential equation with jumps associated with the GST process**

The generator $\widetilde{L} = -\widetilde{H}$ of the ground state-transformed process $(\widetilde{X}_t)_{t\geq 0}$ can be determined explicitly giving (1.3), which has first appeared in [23]. Since H and $\widetilde{H}$ are unitary equivalent by (2.8), we have $\operatorname{Dom}(\widetilde{H}) = U \operatorname{Dom}(H)$, and since H is closed, also $\widetilde{H}$ is a closed operator. Moreover, since H is self-adjoint, $\widetilde{H}$ is also self-adjoint with core $C_{\rm c}^\infty(\mathbb{R}^d)$.

To study the multifractal spectrum, first we show the existence of a solution to the martingale problem for $(\widetilde{L}, C_{\rm c}^2(\mathbb{R}^d))$, and provide a jump SDE representation for the ground state-transformed process, which we call the *ground state SDE*. We write $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$ for a shorthand notation.

For the ground state SDE we will use the following condition (see also Remark 3.1 below).

Assumption 3.1. *Let φ_0 be the ground state of H. We assume that the function $x \mapsto \nabla \ln \varphi_0(x)$, $x \in \mathbb{R}^d$, is locally bounded.*

Theorem 3.1. *Let Assumptions 2.1 and 3.1 hold. Consider the stochastic differential equation with jumps*

$$M_t = M_0 + \sigma B_t + \int_0^t \sigma\nabla \ln \varphi_0(M_s)\,\mathrm{d}s + \int_0^t \int_{|z|\le 1} \frac{\varphi_0(M_s+z)-\varphi_0(M_s)}{\varphi_0(M_s)} z\nu(z)\mathrm{d}z\mathrm{d}s$$
$$+ \int_0^t \int_{|z|\le 1} \int_0^\infty z\mathbf{1}_{\left\{v\le \frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} \widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}v) + \int_0^t \int_{|z|>1} \int_0^\infty z\mathbf{1}_{\left\{v\le \frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}v), \tag{3.1}$$

where $(B_t)_{t\ge 0}$ is an $\mathbb{R}^d$-valued Brownian motion with covariance matrix σ, and N is a Poisson random measure on $[0,\infty)\times\mathbb{R}^d_*\times[0,\infty)$ with intensity $\mathrm{d}t\nu(z)\mathrm{d}z\mathrm{d}v$. The GST process constructed in Theorem 2.1 is a weak solution of (3.1)*.*

Proof. Let

$$\widetilde{X}^f_t = f(\widetilde{X}_t) - f(\widetilde{X}_0) - \int_0^t \widetilde{L}f(\widetilde{X}_r)\mathrm{d}r, \quad t\ge 0, \tag{3.2}$$

where $f\in \mathrm{Dom}(\widetilde{L})$. Using a general result, see e.g. Kurtz [30, Th. 2.3], we have that $(\widetilde{X},\widetilde{\mathbb{P}}^x)$ is a weak solution of the SDE (3.1) if and only if $\widetilde{\mathbb{P}}^x$ solves the $(\widetilde{L}, C^2_{\mathrm{c}})$ martingale problem with initial value $x\in\mathbb{R}^d$, that is, $(\widetilde{X}^f_t)_{t\ge0}$ is a martingale under $\widetilde{\mathbb{P}}^x$, for all $f\in C^2_{\mathrm{c}}(\mathbb{R}^d)$.

Using Assumption 3.1 and that $\varphi_0>0$ is bounded continuous, we see that the functions $x\mapsto \int_{\mathbb{R}^d}\varphi_0(z+x)/\varphi_0(z)(1\wedge|z|^2)\nu(\mathrm{d}z)$, $x\mapsto\int_{|z|\le1} z(\varphi_0(z+x)-\varphi_0(z))/\varphi_0(z)\nu(\mathrm{d}z)$ and $x\mapsto\nabla\log\varphi_0(x)$ are locally bounded, and the conditions in [30, Th. 2.3] hold. Also, since $\widetilde{L}$ is a closed operator, by using a mollifier we can close $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ in the C^2-norm as in [8, Th. 2.37] to obtain that $C^2_{\mathrm{c}}(\mathbb{R}^d)\subset \mathrm{Dom}(\widetilde{L})$.

Let (M,P^x) be a weak solution to (3.1) on a suitable probability space with probability measure P^x, and starting point $P^x(M_0=x)=1$. Write for the drift $b:\mathbb{R}^d\to\mathbb{R}^d$,

$$b(x) = \sigma\nabla\ln\varphi_0(x) + \int_{|z|\le1}\frac{\varphi_0(x+z)-\varphi_0(x)}{\varphi_0(x)} z\nu(z)\mathrm{d}z.$$

Using Itô's formula for $\mathbb{R}^d$-valued semimartingales, see e.g. [17], for any real-valued $f\in C^2_{\mathrm{c}}(\mathbb{R}^d)$ we have

$$\begin{aligned}
f(M_t) \;=\;& f(M_0) + \int_0^t \nabla f(M_{s-})\cdot\mathrm{d}(\sigma B_s) + \int_0^t\nabla f(M_s)\cdot b(M_s)\mathrm{d}s + \frac12\int_0^t \sigma\nabla\cdot\sigma\nabla f(M_s)\mathrm{d}s\\
&+\int_0^t\int_{\mathbb{R}^d_*}\int_0^\infty\left[f\left(M_{s-}+z\mathbf{1}_{\left\{v\le\frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}}\right)-f(M_{s-})\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}v)\\
&+\int_0^t\int_{|z|>1}\int_0^\infty\left[f\left(M_{s-}+z\mathbf{1}_{\left\{v\le\frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}}\right)-f(M_{s-})\right]\mathrm{d}v\nu(z)\mathrm{d}z\mathrm{d}s\\
&+\int_0^t\int_{|z|\le1}\int_0^\infty\Bigg[f\left(M_{s-}+z\mathbf{1}_{\left\{v\le\frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}}\right)-f(M_{s-})\\
&\qquad\qquad -\nabla f(M_{s-})\cdot z\mathbf{1}_{\left\{v\le\frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}}\Bigg]\mathrm{d}v\nu(z)\mathrm{d}z\mathrm{d}s.
\end{aligned}$$

Note that in the second-to-last integral the integrand is zero for all v larger than the ground state ratio $\varphi_0(M_{s-}+z)/\varphi_0(M_{s-})$. It is thus equal to

$$\int_0^t\int_{|z|>1}\left(f(M_{s-}+z)-f(M_{s-})\right)\frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\nu(z)\mathrm{d}z\mathrm{d}s.$$

Similarly, the last integral equals

$$\int_0^t \int_{|z|\leq 1} \big(f(M_{s-}+z) - f(M_{s-}) - z\cdot\nabla f(M_{s-})\big)\frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\nu(z)\mathrm{d}z\mathrm{d}s.$$

Since f has bounded first and second derivatives, the Brownian component and the compensated Poisson integral are martingales, therefore

$$M_t^f = f(M_t) - f(M_0) - \int_0^t \widetilde{L}f(M_s)\mathrm{d}s, \quad t\geq 0,$$

is a P^x-martingale.

It remains to show that the probability measures $P^x = \widetilde{\mathbb{P}}^x$ constructed in Theorem 2.1 solve the $(\widetilde{L}, C_c^2)$ martingale problem with initial value $x\in\mathbb{R}^d$. Consider the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of $(\widetilde{X}_t^f)_{t\geq 0}$ and let $0\leq s\leq t$. Since $\mathbb{E}_{\widetilde{\mathbb{P}}^x}[\widetilde{X}_t^f|\mathcal{F}_s] = \widetilde{X}_s^f + \mathbb{E}_{\widetilde{\mathbb{P}}^x}[\widetilde{X}_t^f - \widetilde{X}_s^f|\mathcal{F}_s]$, we only need to show that the second term vanishes. By the Markov property of $(\widetilde{X}_t)_{t\geq 0}$ established in Theorem 2.1, we have

$$\mathbb{E}_{\widetilde{\mathbb{P}}^x}[f(\widetilde{X}_t)|\mathcal{F}_s] = \widetilde{T}_{t-s}f(\widetilde{X}_s), \quad 0\leq s\leq t.$$

By differentiability of the function $t\mapsto \widetilde{T}_t$ we obtain for all $t\geq 0$ that $\frac{d}{dt}\widetilde{T}_t f = \widetilde{L}\widetilde{T}_t f = \widetilde{T}_t\widetilde{L}f$, and hence $\widetilde{T}_t f - f = \int_0^t \widetilde{L}\widetilde{T}_r f\mathrm{d}r$. Thus we have

$$\begin{aligned}
\mathbb{E}_{\widetilde{\mathbb{P}}^x}\left[f(\widetilde{X}_t) - f(\widetilde{X}_s) - \int_s^t \widetilde{L}f(\widetilde{X}_r)\mathrm{d}r\,|\mathcal{F}_s\right] &= \widetilde{T}_{t-s}f(\widetilde{X}_s) - f(\widetilde{X}_s) - \int_s^t \widetilde{L}\widetilde{T}_{r-s}f(\widetilde{X}_s)\mathrm{d}r \\
&= \widetilde{T}_{t-s}f(\widetilde{X}_s) - f(\widetilde{X}_s) - \int_0^{t-s} \widetilde{L}\widetilde{T}_r f(\widetilde{X}_s)\mathrm{d}r \\
&= \widetilde{T}_{t-s}f(\widetilde{X}_s) - f(\widetilde{X}_s) - \widetilde{T}_{t-s}f(\widetilde{X}_s) + f(\widetilde{X}_s) = 0,
\end{aligned}$$

as required. □

Remark 3.1.

(1) In general little information is available on the regularity of φ_0. In some specific cases of potentials growing to infinity at infinity and the operator $L = (-\mathrm{d}^2/\mathrm{d}x^2)^{1/2}$ it is known that the ground state is analytic [33, 11]. However, it is also known that the ground state for Brownian motion in a finitely deep potential well, i.e., $V(x) = -v\mathbf{1}_{\{|x|\leq a\}}$, $v, a>0$, is only C^1.

(2) The conditions under which the ground state SDE has a solution and Theorem 3.1 holds can be improved. For a large class of Lévy processes $(X_t)_{t\geq 0}$ and potentials V it can be shown that for large enough $|x|$ and suitable constants $C_1, C_2>0$,

$$C_1\frac{\nu(x)}{V(x)} \leq \varphi_0(x) \leq C_2\frac{\nu(x)}{V(x)} \quad \text{for } V(x)\to\infty \ \text{ as } \ |x|\to\infty$$

$$C_1\nu(x) \leq \varphi_0(x) \leq C_2\nu(x) \qquad \text{for } V(x)\to 0 \ \text{ as } \ |x|\to\infty.$$

For precise statements and conditions we refer to [22, 24]. For illustration consider $d=1$, $V(x)\asymp x^{2m}$, $m>0$, and a symmetric α-stable process; then the above implies $\varphi_0(x)\asymp |x|^{-d-\alpha-2m}$, and thus far enough from the origin the drift would become $b(x) = \sigma\frac{\mathrm{d}}{\mathrm{d}x}\ln\varphi_0(x) \asymp -\frac{\mathrm{sgn}(x)}{|x|}$. Hence for large enough values of X_t we get $X_t b(X_t) < 0$, and this pull-back mechanism would prevent the paths from exploding. Since our main concern here is the multifractal behaviour of GST processes, the ground state SDE will be studied in further detail elsewhere.

(3) We note that we are not concerned with uniqueness of the solution of the martingale problem. Since we show below that any solution of the SDE (3.1) has the same multifractal nature, we only need to know that the GST process is a solution.

3.2. Multifractal spectrum of GST processes

Now we are in the position to state and prove the multifractal nature of local Hölder exponents of ground state-transformed processes via the ground state SDE. Recall the notations in (1.7)-(1.8).

Theorem 3.2. *Let Assumptions 2.1 and 3.1 hold, and $(M, \mathbb{P}^x)$ be a weak solution of (3.1).*

(1) *If $\sigma \neq 0$, i.e., the underlying Lévy process has a Brownian component, then almost surely*

$$D_M(h) = D^1_X(h), \quad h > 0.$$

(2) *If $\sigma = 0$, i.e., the underlying Lévy process is a pure jump process, and*
   (i) *either $\beta_\nu \in [1, 2]$,*
   (ii) *or $\beta_\nu \in (0, 1)$ with $\varphi_0 \in C^{k+1}(\mathbb{R}^d)$ and $k \geq 1/\beta_\nu$,*

   then almost surely

$$D_M(h) = D^2_X(h), \quad h > 0.$$

In the remainder of this section we prove this theorem through a sequence of auxiliary results.

3.2.1. Associated Poisson point process

Recall that the jumps of the Poisson measure N give rise to a Poisson point process with measure $\nu(z)\mathrm{d}z$. The pointwise regularity of Lévy processes with infinite jump measure, i.e., $\int_{\mathbb{R}^d} \nu(z)\mathrm{d}z = \infty$, relies on a configuration of a dense set of jumps. For the process $(M_t)_{t\geq 0}$ the jump configuration is more involved as it depends on the entire paths of the process. Indeed, the indicator factor in the compensated Poisson integral in (3.1) implies that our process jumps only as long as the ground state ratio is not too small. Thus $(M_t)_{t\geq 0}$ jumps less often than the underlying Lévy process. We prove that the underlying Poisson point process characterizes the pointwise regularity of $(M_t)_{t\geq 0}$.

Let $N(\mathrm{d}t, \mathrm{d}z, \mathrm{d}v)$ be a Poisson measure with intensity $\mathrm{d}t\, n(\mathrm{d}z, \mathrm{d}v)$ on $\mathbb{R}^+ \times E$ with $E = \mathbb{R}^d_* \times (0, \infty)$, endowed with the product Borel σ-field $\mathcal{B}(E)$. Let $\{E_k, k \in \mathbb{N}_*\}$ be a partition of E with $E_k \in \mathcal{B}(E)$ and $n(E_k) < \infty$. It is well-known [15, Ths 8.1, 9.1] that there exists

- a sequence of exponential random variables $\{\tau^{(k)}_i, i \in \mathbb{N}\}$ with parameter $n(E_k)$,
- a sequence of random variables $\{\xi^{(k)}_i, i \in \mathbb{N}\}$ with distribution $\mathbf{1}_{E_k} n(\mathrm{d}z, \mathrm{d}v)/n(E_k)$,

such that

$$N((0, t] \times U) = |\{s \in D : s \leq t,\, p(s) \in U\}|, \quad \text{for all } t > 0,\, U \in \mathcal{B}(E),$$

where p is the point process defined by

$$p\left(\sum_{\ell=0}^{i} \tau^{(k)}_\ell\right) = \xi^{(k)}_i, \quad k, i = 1, 2, \ldots$$

and

$$D = \bigcup_{k=0}^{\infty} \left\{ \sum_{\ell=0}^{i} \tau^{(k)}_\ell : i \in \mathbb{N} \right\}.$$

Here, all $\tau^{(k)}_i, \xi^{(k)}_i$ are mutually independent random variables on the same probability space. Extending the probability space, if necessary, by passing to a product probability space, we can find a sequence of uniform random variables η_i in $[0,1]$ that is independent of p. Define $p' : D \to E \times [0,1]$ by

$$p'\left(\sum_{\ell=0}^{i} \tau_\ell^{(k)}\right) = (\xi_i^{(k)}, \eta_i) \quad k, i = 1, 2, \dots$$

It can be shown [15, Ths 8.1, 9.1] that the counting measure

$$N_{p'}((0,t] \times U \times I) = \left|\left\{s \in D : s \le t,\, p'(s) \in U \times I\right\}\right|, \quad \text{for all } t > 0,\, U \in \mathcal{B}(E),\, I \in \mathcal{B}([0,1]),$$

is also a Poisson measure, with intensity $\mathrm{d}t\, n(\mathrm{d}z, \mathrm{d}v)\mathbf{1}_{[0,1]}(x)\mathrm{d}x$. In particular, almost surely,

$$N((0,t] \times U) = N_{p'}((0,t] \times U \times [0,1]). \tag{3.3}$$

From now on, we consider the Poisson measure N as part of the weak solution $(M, \widetilde{\mathbb{P}}^x)$ of the SDE (3.1) on a probability space. Possibly on the extended probability space, we have $N_{p'}$ satisfying (3.3) which will serve as an auxiliary measure to prove a covering property satisfied by a family of point systems induced by the jumps of our process $(M_t)_{t\ge 0}$. Informally, the measure $N_{p'}$ allows us to remove spatial dependence of the jump kernel; a similar argument has been first used in [42]. In what follows we will write

$$p = \{(s, z(s), v(s)) : s \in D\} \quad \text{and} \quad p' = \{(s, z(s), v(s), x(s)) : s \in D\}.$$

3.2.2. **Hölder regularity**

First we determine the pointwise Hölder exponent of the sample paths of the ground state SDE under the assumption that the ratio of ground state evaluations appearing in the coefficients of the SDE is bounded both from below and above, i.e., we assume that there exists $0 < c < 1$ such that

$$c \le \frac{\varphi_0(x+z)}{\varphi_0(x)} \le 1/c, \quad x \in \mathbb{R}^d, |z| \le 1. \tag{3.4}$$

In a next step we remove this constraint by using a localization argument to get the result in a desirable generality.

The following general result is due to Jaffard [19, Lem. 1], which is essential in deriving an upper bound for the Hölder exponent of a locally bounded function with a dense set of jump discontinuities.

Lemma 3.1. *Let $f : \mathbb{R} \to \mathbb{R}^d$ be a càdlàg function having a dense set of jump discontinuities of size z_n at the time-points t_n. Then for every $t \in \mathbb{R}$ and every sequence of jump discontinuities $t_{n_k} \to t$ as $k \to \infty$, we have*

$$H_f(t) \le \liminf_{k\to\infty} \frac{\ln z_{n_k}}{\ln |t - t_{n_k}|}.$$

Clearly, only the small jumps have an impact on the local regularity. Write

$$J = \left\{ s \ge 0 : |z(s)| \le 1,\, v(s) \le \frac{\varphi_0(M_{s-} + z(s))}{\varphi_0(M_{s-})} \right\}.$$

By the properties of the (compensated) Poisson integral, the solution to the ground state SDE (3.1) makes a jump at each $s \in J$, of size $|z(s)|$. Borrowing an idea from [19], we consider a family of limsup sets built from the Poisson point process p. Recall that β_ν is the Blumenthal-Getoor index of the Lévy measure $\nu(z)\mathrm{d}z$ defined in (1.6). For all $\delta > 0$, define

$$A(\varepsilon, \delta) = \bigcup_{s\in J,\ |z(s)|\le\varepsilon} (s - |z(s)|^{\beta_\nu \delta}, s + |z(s)|^{\beta_\nu \delta}),$$

and

$$A(\delta) = \limsup_{\varepsilon \downarrow 0} A(\varepsilon, \delta). \tag{3.5}$$

This family of sets satisfies a convenient covering property when $\delta < 1$.

Lemma 3.2. *For all $\delta < 1$ we have $A_\delta = [0, \infty)$, almost surely.*

Proof. Define

$$J' = \left\{ s \geq 0 : |z(s)| \leq 1,\, v(s) \leq \frac{\varphi_0(M_{s-} + z(s))}{\varphi_0(M_{s-})},\, x(s) \leq \frac{c\varphi_0(M_{s-})}{\varphi_0(M_{s-} + z(s))} \right\} \subset J,$$

and A'_δ as in (3.5) with J replaced by J'. Observe that by the lower bound in (3.4), the right hand side of the bound concerning $x(s)$ in the above set is a random number in $[0, 1]$. Since $A'_\delta \subset A_\delta$ for all $\delta \geq 0$, it remains to show that for any fixed $\delta < 1$, we have $A'_\delta = [0, \infty)$ almost surely. The result then follows by the monotonicity of the sets A_δ in δ.

Step 1: First we note that the counting measure

$$\mu(\mathrm{d}s, \mathrm{d}y) = \sum_{s \in J} \delta_{(s, |z_s|^{\delta\beta_\nu})}$$

is a Poisson random measure with intensity $\mathrm{d}s(c\pi_\delta(\mathrm{d}y))$ on $\mathbb{R}^+ \times (0, 1]$, where π_δ is the image measure of $\nu(z)\mathrm{d}z 1_{|z| \leq 1}$ by the map $z \mapsto |z|^{\delta\beta_\nu}$ and c is the constant in (3.4). For any predictable non-negative process $(s, y) \mapsto H(s, y)$,

$$\begin{aligned}
&\int_0^t \int_0^1 H(s, y)\mu(\mathrm{d}s, \mathrm{d}y) - \int_0^t \int_{|z| \leq 1} H(s, |z(s)|^{\delta\beta_\nu}) c\nu(z)\mathrm{d}z\mathrm{d}s \\
&= \int_0^t \int_{|z| \leq 1} \int_0^\infty \int_0^1 \mathbf{1}_{\left\{ v(s) \leq \frac{\varphi_0(M_{s-} + z(s))}{\varphi_0(M_{s-})},\, x(s) \leq \frac{c\varphi_0(M_{s-})}{\varphi_0(M_{s-} + z(s))} \right\}} H(s, |z(s)|^{\delta\beta_\nu}) \widetilde{N}_{p'}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}v, \mathrm{d}x)
\end{aligned}$$

is a local martingale. Then the compensator of μ is $c\,\mathrm{d}t\,\pi_\delta(\mathrm{d}y)$. By [17, Ch.2, Th.1.8], μ is a Poisson measure with intensity $c\,\mathrm{d}t\,\pi_\delta(\mathrm{d}y)$.

Step 2: Applying the integral test of covering for limsup sets built from a Poisson measure, see [4, 39], we only need to show that

$$\int_0^1 \exp\left(2\int_t^1 c\pi_\delta((y, 1))\mathrm{d}y\right)\mathrm{d}t = \infty.$$

The divergence of this integral can be proved by a modification of [19, Lem. 2]. Note that

$$\int_t^1 c\pi_\delta((y, 1))\mathrm{d}y = c\int_{t^{\frac{1}{\delta\beta_\nu}}}^1 \left(\int_{u < |x| < 1} \nu(x)\mathrm{d}x\right) \delta\beta_\nu u^{\delta\beta_\nu - 1}\mathrm{d}u.$$

Write $C_j = \int_{2^{-j-1} < |x| \leq 2^{-j}} \nu(x)\mathrm{d}x$ and $\omega(u) = \int_{u < |x| < 1} \nu(x)\mathrm{d}x$. Let $j(t)$ be the unique integer such that $2^{j(t)-3} < t^{\frac{1}{\delta\beta_\nu}} \leq 2^{-j(t)-2}$. Then we have

$$\begin{aligned}
\int_{t^{\frac{1}{\delta\beta_\nu}}}^1 \omega(u)\delta\beta_\nu u^{\delta\beta_\nu - 1}\mathrm{d}u &\geq \int_{2^{j(t)-2}}^{2^{j(t)-1}} \omega(u)\delta\beta_\nu u^{\delta\beta_\nu - 1}\mathrm{d}u \\
&\geq C_{j(t)}\delta\beta_\nu (2^{-j(t)-2})^{\delta\beta_\nu - 1} 2^{-j(t)-2} \\
&= C_{j(t)}\delta\beta_\nu (2^{-j(t)-2})^{\delta\beta_\nu}.
\end{aligned}$$

By the definition of β_ν, for any $r \in (\delta\beta_\nu, \beta_\nu)$, there exist infinitely many j such that $C_j \geq 2^{rj}$. For any such j we have

$$\int_{2^{-(j+3)\delta\beta_\nu}}^{2^{-(j+2)\delta\beta_\nu}} \exp\left(2\int_t^1 c\pi_\delta((y,1))\mathrm{d}y\right)\mathrm{d}t \geq (2^{\delta\beta_\nu}-1)2^{-(j+3)\delta\beta_\nu}\exp(c\delta\beta_\nu 2^{1-2\delta\beta_\nu}2^{j(r-\delta\beta)}),$$

which is bounded from below by 1 for all j sufficiently large. □

The latter lemma is a uniform approximation property of every time by the jumps. It is clear that A_δ is monotone in δ, while the critical value is $\delta = 1$ for which the limsup set may or may not cover the semi-axis. As soon as $\delta < 1$, full covering occurs. In particular, whenever $\delta < 1$, for every $t \geq 0$ there exist infinitely many $s_n \in J$ with $|z(s_n)| \downarrow 0$ such that

$$|t - s_n| \leq |z(s_n)|^{\beta_\nu\delta}.$$

For fixed time-points, one might expect an improved inequality to hold, which motivates the notion of the pointwise approximation rate defined below.

Definition 3.1. *Let $(t_n, r_n) \in \mathbb{R}^+ \times \mathbb{R}^*$ be a family of points. We call*

$$\delta_t = \sup\left\{\delta \geq 0 : |t - t_n| \leq r_n^{\beta_\nu\delta} \text{ infinitely often }\right\} \tag{3.6}$$

the approximation rate *of $t \in \mathbb{R}^+$ by the family of points.*

The approximation rate is crucial in investigating the pointwise Hölder exponent of jump processes. By the covering lemma, for all $t \geq 0$ we have $\delta_t \geq 1$, almost surely. The use of this concept will appear clearly in the upper estimate of $H_M(t)$ below.

Proposition 3.1. *For all $t \geq 0$,*

$$H_M(t) \leq \frac{1}{\beta_\nu\delta_t}$$

almost surely.

Proof. Take any $t \in A_\delta$. An application of Lemma 3.1 to $(M_t)_{t\geq 0}$ and the set of s_n in (3.6) implies that $H_M(t) \leq 1/(\beta_\nu\delta)$. For an arbitrary t we have the following cases. If $\delta_t < \infty$, then for any $\varepsilon > 0$, $t \in A_{\delta_t - \varepsilon}$, we have $H_M(t) \leq 1/(\beta_\nu(\delta_t - \varepsilon))$. Letting $\varepsilon \to 0$ gives the result. If $\delta_t = \infty$, then $t \in \cap_{\delta\geq 1} A_\delta$, thus $H_M(t) = 0$, which is the claimed upper bound. □

To derive a lower bound, we need to control the increments of the sample paths. An analogue of the following result appears in [1] for Lévy processes, however, since the GST processes have position-dependent increments, we need a substantial upgrading. For each $n \in \mathbb{N}$, write

$$Y_n(t) = \int_0^t \int_{|z|\leq 2^{-\frac{n}{\delta\beta_\nu}}} \int_0^\infty \mathbf{1}_{\left\{v \leq \frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}v).$$

Lemma 3.3. *Let $\delta > 1$. There exist finite constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{P}\left(\sup_{s,t\in[0,1],\ |s-t|\leq 2^{-n}} |Y_n(t) - Y_n(s)| \geq 3n2^{-\frac{n}{\delta\beta_\nu}}\sqrt{d}\right) \leq c_1 e^{-c_2 n}.$$

Proof. Let $I_{n,k} = [k2^{-n}, (k+1)2^{-n})$. Using a dyadic approximation, the required probability can be bounded from above by

$$\sum_{k=0}^{2^n-1} \mathbb{P}\left(\sup_{t\in I_{n,k}} |Y_n(t) - Y_n(k2^{-n})| \geq n2^{-\frac{n}{\delta\beta_\nu}}\sqrt{d}\right). \tag{3.7}$$

We estimate the sum term by term. For each k, consider the semimartingale

$$\widetilde{Y}_n(t) = 2^{\frac{n}{\delta\beta_\nu}} \left(Y_n(t+k2^{-n}) - Y_n(k2^{-n})\right), \quad t \in I_{n,k}.$$

Applying Itô's formula with the map $x \mapsto e^{x\cdot\xi}$, where $\xi = \vec{e_i}$ is the canonical orthonormal basis of $\mathbb{R}^d$, we obtain

$$\begin{aligned} e^{\widetilde{Y}_n(t)\cdot\xi} \;=\; 1 &+ \int_{k2^{-n}}^{t}\int_{|z|\le 2^{-\frac{n}{\delta\beta_\nu}}}\int_0^\infty e^{\widetilde{Y}_n(s-)\cdot\xi}\left(\exp\left(2^{\frac{n}{\delta\beta_\nu}}\mathbf{1}_{\left\{v\le\frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} z\cdot\xi\right)-1\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}v) \\ &+ \int_{k2^{-n}}^{t}\int_{|z|\le 2^{-\frac{n}{\delta\beta_\nu}}}\int_0^\infty e^{\widetilde{Y}_n(s-)\cdot\xi}\left(\exp\left(2^{\frac{n}{\delta\beta_\nu}}\mathbf{1}_{\left\{v\le\frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} z\cdot\xi\right)-1\right. \\ &\qquad\left. -2^{\frac{n}{\delta\beta_\nu}}\mathbf{1}_{\left\{v\le\frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} z\cdot\xi\right)\mathrm{d}v\nu(z)\mathrm{d}z\mathrm{d}s, \end{aligned}$$

for all $t \in I_{n,k}$. Define the stopping times $\tau_r = \inf\{t \in I_{n,k} : |\widetilde{Y}_n(t)| \ge r\}$, $r \in \mathbb{N}$, with the convention that $\inf\emptyset = \infty$. By the càdlàg property of sample paths, $\tau_r \to \infty$ as $r \to \infty$, almost surely. Since the stopped compensated Poisson integral is a centered martingale, on taking expectation in the above formula and using $|e^u - 1 - u| \le u^2$ for $|u| \le 1$, we get

$$\begin{aligned} \mathbb{E}[e^{\widetilde{Y}_n(t\wedge\tau_r)\cdot\xi}] &\le 1 + \mathbb{E}\Big[\int_{k2^{-n}}^{t\wedge\tau_r}\int_{|z|\le 2^{-\frac{n}{\delta\beta_\nu}}}\int_0^\infty e^{\widetilde{Y}_n(s-)\cdot\xi}\mathbf{1}_{\left\{v\le\frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}}\left(2^{\frac{n}{\delta\beta_\nu}} z\cdot\xi\right)^2\mathrm{d}v\nu(z)\mathrm{d}z\mathrm{d}s\Big] \\ &= 1 + \mathbb{E}\Big[\int_{k2^{-n}}^{t\wedge\tau_r}\int_{|z|\le 2^{-\frac{n}{\delta\beta_\nu}}}\frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}e^{\widetilde{Y}_n(s-)\cdot\xi}\left(2^{\frac{n}{\delta\beta_\nu}} z\cdot\xi\right)^2\nu(z)\mathrm{d}z\mathrm{d}s\Big]. \end{aligned}$$

Using the upper bound in (3.4), we furthermore obtain

$$\mathbb{E}[e^{\widetilde{Y}_n(t\wedge\tau_r)\cdot\xi}] \le 1 + \frac{1}{c}\mathbb{E}\Big[\int_{k2^{-n}}^{t\wedge\tau_r} e^{\widetilde{Y}_n(s-)\cdot\xi}\mathrm{d}s\int_{|z|\le 2^{-\frac{n}{\delta\beta_\nu}}} 2^{\frac{2n}{\delta\beta_\nu}}|z|^{2-\delta\beta_\nu}|z|^{\delta\beta_\nu}\nu(z)\mathrm{d}z\Big].$$

The integral over z in the expectation is bounded above by

$$2^n\int_{|z|\le 1}|z|^{\delta\beta_\nu}\nu(z)\mathrm{d}z = C_1 2^n,$$

with a suitable constant C_1 which does not depend on n and is finite since $\delta > 1$. By Fubini's theorem,

$$\mathbb{E}[e^{\widetilde{Y}_n(t\wedge\tau_r)\cdot\xi}] \le 1 + \frac{2^n}{c_1}\int_{k2^{-n}}^{t}\mathbb{E}[e^{\widetilde{Y}_n(s\wedge\tau_r)\cdot\xi}]\,\mathrm{d}s,$$

where $C_2 = C_1/c$. Gronwall's lemma yields then

$$\mathbb{E}[e^{\widetilde{Y}_n(t\wedge\tau_r)\cdot\xi}] \le e^{(t-k2^{-n})2^n/C_2} \le e^{1/C_2},$$

for all $t \in I_{n,k}$. Letting $r \to \infty$ and using Fatou's lemma, we get $\mathbb{E}[e^{\widetilde{Y}_n(t)\cdot\xi}] \le e^{1/C_2}$, and similarly $\mathbb{E}[e^{-\widetilde{Y}_n(t)\cdot\xi}] \le e^{1/C_2}$. Hence, using that ${x_1}^2 + \cdots + {x_d}^2 \le d\max_i x_i^2$, we have

$$\mathbb{E}[e^{|\widetilde{Y}_n(t)|/\sqrt{d}}] \le \sum_{i=1}^d \mathbb{E}[e^{|\widetilde{Y}_n(t)\cdot\vec{e_i}|}] \le 2de^{1/C_2}.$$

To conclude, by the Markov inequality we see that each term in (3.7) is bounded from above by $e^{-n}\mathbb{E}[e^{|\widetilde{Y}_n(t)|/\sqrt{d}}]$, which is summable in n. □

Remark 3.2. This lemma can be extended to any bounded interval. We thus focus on the unit interval $[0,1]$.

We can now prove a lower bound for the Hölder exponent of the compensated Poisson integral

$$Y_t = \int_0^t \int_{|z|\le 1} \int_0^\infty \mathbf{1}_{\left\{v \le \frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} z \widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}v).$$

Proposition 3.2. *For all $t \in [0,1] \setminus J$,*

$$H_Y(t) \ge \frac{1}{\delta_t \beta_\nu},$$

almost surely.

Proof. The Borel-Cantelli lemma combined with Lemma 3.3 give that for all n larger than a suitable $n_0 \in \mathbb{N}$,

$$\sup_{s,t\in[0,1],\,|s-t|\le 2^{-n}} |Y_n(t) - Y_n(s)| \le 3\sqrt{d} n 2^{-\frac{n}{\delta\beta_\nu}}, \quad \text{a.s.}$$

Fix a point of continuity $t \in [0,1] \setminus A_\delta$. Let s be close enough to t so that for some $n < n_0$,

$$2^{-n-1} < |t-s| \le 2^{-n}.$$

Then

$$|Y_n(t) - Y_n(s)| \le 6\sqrt{d} \log\left(\frac{1}{|s-t|}\right) |t-s|^{\frac{1}{\delta\beta_\nu}}.$$

Enlarging the value of n_0 if necessary, we see that $t \notin A_\delta$ implies that any jump $s_p \in [s,t]$ satisfies

$$2^{-n} \ge |s-t| \ge |s_p - t| \ge |z(s_p)|^{\delta\beta_\nu},$$

i.e., there are no jumps at the time-points $s_p \in J \cap [s,t]$ of size $|z(s_p)| \ge 2^{-\frac{n}{\delta\beta_\nu}}$. Hence,

$$\begin{aligned}
&\left|\int_s^t \int_{1\ge|z|\ge 2^{-\frac{n}{\delta\beta_\nu}}} \int_0^\infty \mathbf{1}_{\left\{v \le \frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} z \widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}v)\right| \\
&\quad = \left|\int_s^t \int_{1\ge|z|\ge 2^{-\frac{n}{\delta\beta_\nu}}} \int_0^\infty \mathbf{1}_{\left\{v \le \frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} z \mathrm{d}v \nu(z) \mathrm{d}z \mathrm{d}s\right| \\
&\quad \le |s-t| \frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})} \int_{1\ge|z|\ge 2^{-\frac{n}{\delta\beta_\nu}}} |z| \nu(z) \mathrm{d}z.
\end{aligned}$$

The integral over z is bounded above by

$$(2^{-\frac{n}{\delta\beta_\nu}})^{1-\delta\beta_\nu} \int_{|z|\le 1} |z|^{\delta\beta_\nu} \nu(z) \mathrm{d}z \le C|s-t|^{\frac{1}{\delta\beta_\nu}-1},$$

with a constant $C > 0$. Combining these estimates, we get

$$\begin{aligned}
|Y_s - Y_t| &\le |Y_n(t) - Y_n(s)| + \left|\int_s^t \int_{1\ge|z|\ge 2^{-\frac{n}{\delta\beta_\nu}}} \int_0^\infty \mathbf{1}_{\left\{v \le \frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} z \widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}v)\right| \\
&\le c|t-s|^{\frac{1}{\delta\beta_\nu}} \log \frac{1}{|s-t|},
\end{aligned}$$

where c is a finite constant dependent on M and d. Hence, almost surely, for all rational $\delta > 1$ we have $H_Y(t) \ge 1/(\delta\beta_\nu)$ at all times of continuity $t \in [0,1] \setminus A_\delta$. By the definition of δ_t, it is seen that $H_Y(t) \ge 1/(\delta_t\beta_\nu)$, for all continuity points $t \in [0,1]$, almost surely. □

Remark 3.3. Using the argument in the proof of Proposition 3.1, we can similarly show $H_Y(t) \le 1/(\delta_t\beta_\nu)$ for all t.

Theorem 3.3. *Under the assumptions of Theorem 1.3, for all times of continuity* t,

$$H_M(t) = \begin{cases} \frac{1}{\delta_t \beta_\nu} \wedge \frac{1}{2} & \text{if } \sigma \neq 0 \\ \\ \frac{1}{\delta_t \beta_\nu} & \text{if } \sigma = 0 \end{cases}$$

almost surely.

Proof. We distinguish three situations according to the matrix σ and the value of β_ν.

Case 1: Let $\sigma \neq 0$ and $\beta_\nu \in (0, 2]$. Recall that for any $f, g : \mathbb{R}^+ \to \mathbb{R}^d$ locally bounded functions $H_{f+g}(t) \geq \min(H_f(t), H_g(t))$, where equality holds when the Hölder exponents of f and g are different at t. Since the Hölder exponent of Brownian motion is $1/2$ everywhere and the drift terms are differentiable at every t (necessarily their Hölder exponent is larger or equal to 1), we see that the sum of Brownian motion and the two drifts has Hölder exponent equal to $1/2$ everywhere. The uncompensated Poisson integral is locally constant, thus it does not influence the local regularity of $(M_t)_{t\geq 0}$ except on the set of jump times (finite in any bounded interval). The compensated Poisson integral has Hölder exponent $1/(\delta_t\beta_\nu)$ at any point of continuity t. The claimed formula follows if $1/(\delta_t\beta_\nu) \neq 1/2$, otherwise $1/2$ is a straightforward lower bound for $H_M(t)$, and it is also an upper bound due to Lemma 3.1. Thus the identity follows.

Case 2: Let $\sigma = 0$ and $\beta_\nu \in [1, 2]$. In this case, $1/(\delta_t\beta_\nu) \leq 1$, since $\delta_t \geq 1$ for all t, i.e., the drifts are all smoother than the compensated Poisson integral. The result follows.

Case 3: Let $\sigma = 0$ and $\beta_\nu \in (0, 1)$. Our assumption implies that the drift terms are smoother than the compensated Poisson integral. To see this, note that for any locally bounded $f : \mathbb{R}^+ \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathbb{R}^d$, with $F(t) = \int_0^t g(f(s))\mathrm{d}s$, we have that whenever $g \in C^k(\mathbb{R}^d)$ with $k \geq H_f(t)$, it follows that $H_F(t) \geq 1 + H_f(t)$. In particular, we have $H_F(t) > H_f(t)$. Applying this to $f = M$ and with g chosen to be the drift coefficient in (3.1), combined with the fact that $H_M(t) \leq 1/(\delta_t\beta_\nu) \leq 1/\beta_\nu$, yields $H_M(t) = H_Y(t)$, as claimed.

To complete the proof, in a concluding step we remove condition (3.4). Let

$$\Omega_{K,b} = \left\{ \omega \in \Omega : \sup_{t \leq b} |M_t(\omega)| \leq K \right\}.$$

The càdlàg properties of the sample paths imply that $\mathbb{P}(\Omega_{K,b}) \to 1$ as $K \to \infty$. Assumption 2.1 implies that the two-sided inequality in (3.4) holds uniformly for $|z| \leq 1$, $|x| \leq K$, for every $K \in \mathbb{N}_*$, with c dependent on the value of K. For paths in $\Omega_{K,b}$, we have shown the result above. Letting $K \to \infty$, then $b \to \infty$ completes the proof. □

3.2.3. **Proof of Theorem 3.2: multifractal spectrum**

We determine the multifractal spectrum under condition (3.4), the extension to the general situation can be done as at the end of the last subsection.

Note that by Theorem 3.3 it suffices to consider $h \in [0, 1/\beta_\nu]$. For every such h, we have that

$$E_M(h) = \left\{ t \geq 0 : \delta_t = \frac{1}{h\beta_\nu} \right\} \setminus J = \left(\bigcap_{\alpha < 1/(h\beta_\nu)} A_\alpha \right) \setminus \left(\bigcup_{\alpha > 1/(h\beta_\nu)} A_\alpha \right) \setminus J$$

$$= \left(\bigcap_{n \geq 1} A_{1/(h\beta_\nu) - 1/n} \right) \setminus \left(\bigcup_{n \neq 1} A_{1/(h\beta_\nu) + 1/n} \right) \setminus J. \tag{3.8}$$

First we give an upper bound on the Hausdorff dimension of the family of sets $\{A_\delta, \delta \geq 1\}$. Observe that for any j_0,

$$A_\delta \subset \bigcup_{j \geq j_0} \bigcup_{\substack{s \in J \\ 2^{-j-1} \leq |z(s)| < 2^{-j}}} (s - |z(s)|^{\beta_\nu \delta}, s + |z(s)|^{\beta_\nu \delta}).$$

We can use these intervals as a covering system of A_δ. It suffices to show that for any $s > 1/\delta$, almost surely,

$$\sum_{j \geq j_0} (2^{-j\beta_\nu \delta})^s N([0,1] \times [0,1/c] \times \{z : 2^{-j-1} \leq |z| < 2^{-j}\}) < \infty, \tag{3.9}$$

where c is the constant in (3.4). This implies that for all $\delta \geq 1$ we have $\dim_{\mathsf{H}} A_\delta \leq 1/\delta$, almost surely.

Next we prove (3.9). Note that $N_j := N([0,1] \times [0,1/c] \times \{z : 2^{-j-1} \leq |z| < 2^{-j}\})$ is a Poisson random variable with parameter C_j/c, where $C_j = \int_{2^{-j-1} < |z| \leq 2^{-j}} \nu(z)\mathrm{d}z$ as in the previous section. Let $r \in (\beta_\nu, \beta_\nu \delta s)$. Then by the definition of β_ν we have $C_j \leq 2^{jr}$, for all j large enough. Hence by the Markov inequality,

$$\mathbb{P}(N_j \geq 2 \cdot 2^{jr}) \leq \mathbb{P}(|N_j - C_j| \geq 2^{jr}) \leq 2^{-jr}.$$

It then follows by the Borel-Cantelli lemma that $N_j \leq 2^{jr}$ almost surely, for all j sufficiently large. The convergence of the series follows, since we choose $r < \beta_\nu \delta s$.

The above combined with (3.8) implies that $\dim_{\mathsf{H}} E_M(h) \leq \frac{1}{\beta_\nu \delta}$. To obtain a lower bound on the spectrum, we make use of the following general result; for a proof see [20]. Let $|\cdot|$ denote Lebesgue measure in $\mathbb{R}$.

Theorem 3.4. *Let $(\lambda_n, \varepsilon_n)$ be a family of points, with $\lambda_n \in [0,1]$ and $\varepsilon_n > 0$. Define $G_\delta = \limsup_{n \to \infty} (\lambda_n - \varepsilon_n^\delta, \lambda_n + \varepsilon^\delta)$. If $|G_1| = 1$, then for all $\delta \geq 1$*

$$\mathcal{H}^{\phi_\delta}(G_\delta) > 0,$$

where $\phi_\delta(x) = x^{1/\delta} |\log x|^2$, and $\mathcal{H}^{\phi_\delta}(E)$ is the Hausdorff measure of the set E with respect to the gauge function ϕ_δ.

In order to apply the above result, one would need to prove the almost covering $|A_1 \cap [0,1]| = 1$ at critical index $\delta = 1$, which holds for regular Lévy measures such as the isotropic α-stable case with $\nu(z) = |z|^{-\alpha-d}$. However, for some ill-behaved Lévy measures the situation $|A_1 \cap [0,1]| < 1$ may occur. To overcome this, we use a trick from [42, Prop. 3.2] to construct a family of limsup sets A_δ^* embedded in A_δ in the sense that for every $\delta' < \delta$,

$$A_\delta \subset A_\delta^* \subset A_{\delta'} \tag{3.10}$$

satisfying

$$|A_1^* \cap [0,1]| = 1. \tag{3.11}$$

Using (3.10), we can express $E_M(h)$ in (3.8) with all A_δ replaced by A_δ^*. A use of (3.11) then implies that $\mathcal{H}^{\phi_\delta}(A_\delta^*) > 0$ almost surely, for all $\delta \geq 1$. Recalling that $\dim_{\mathsf{H}} A_\delta^* \leq 1/\delta$, we have $\mathcal{H}^{\phi_{1/(h\beta_\nu)}}(A^*_{1/(h\beta_\nu)+1/n}) = 0$, which gives that $\mathcal{H}^{\phi_{1/(h\beta_\nu)}}(E_M(h)) > 0$. This proves that $D_M(h) \geq \beta_\nu h$ almost surely, simultaneously for all $0 \leq h \leq 1/\beta_\nu$, as required.

To complete the argument, it remains to construct the sets A_δ^* satisfying (3.10)-(3.11). For any integers $m < n \le \infty$, let

$$A_\delta^{m,n} = \bigcup_{\substack{s\in J \\ 2^{-n}\le |z(s)|<2^{-m}}} (s - |z(s)|^{\beta_\nu \delta}, s + |z(s)|^{\beta_\nu \delta}).$$

Set $m_1 = 1$. Due to Lemma 3.2, there exists $m_2 > m_1$ such that $[0,1] \subset A_{1-1/2} \subset A_{1-1/2}^{m_1,\infty}$ and $|A_{1-1/2}^{m_1,m_2}| \ge 1/2$. Similarly, there exists $m_3 > m_2$ such that $[0,1] \subset A_{1-1/3} \subset A_{1-1/3}^{m_2,\infty}$ and $|A_{1-1/3}^{m_2,m_3}| \ge 1-1/3$. We define a sequence $(m_j)_{j\ge 1}$ inductively such that for all $j \ge 2$, $|A_{1-1/j}^{m_{j-1},m_j}| \ge 1 - 1/j$. Hence,

$$\Big|\limsup_{j\to\infty} A_{1-1/j}^{m_{j-1},m_j} \cap [0,1]\Big| \ge \limsup_{j\to\infty} \Big|A_{1-1/j}^{m_{j-1},m_j} \cap [0,1]\Big| = 1.$$

Define $A_\delta^* = \limsup_{j\to\infty} A_{\delta(1-1/j)}^{m_j,m_{j+1}}$; then the above formula shows (3.11). To show property (3.10), note that $A_\delta = \limsup_{j\to\infty} A_\delta^{m_j,m_{j+1}}$, and the first inclusion then follows. The second inclusion also holds since for any $\delta' < \delta$ we have $\delta' < \delta(1-1/j)$, for all sufficiently large j. This completes the proof.

3.3. **Concluding remarks**

As seen from the proof, the only case when we require some extra smoothness condition for the ground state is when the Blumenthal-Getoor index is $\beta_\nu < 1$ and $\sigma = 0$. As said in Remark 3.1 this is known to hold in some cases, and it can be expected further to hold more widely.

We conjecture that the multifractal nature of a GST process will change if the ground state is less regular and $\nabla \ln \varphi_0$ is C^ε, with $1 + \varepsilon < 1/\beta_\nu$. To see this, consider a simple representation for a GST process when $\beta_\nu < 1$ and $\sigma = 0$. In such cases the process has finite variation, thus the compensated Poisson integral can be decomposed as a difference of an uncompensated Poisson integral and a drift term. More precisely, the GST process is a weak solution of the simple SDE with jumps

$$M_t = M_0 + \int_0^t b(M_s)\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d_*} z N(\mathrm{d}s, \mathrm{d}z),$$

where N is a Poisson measure with intensity $\mathrm{d}t\nu(z)\mathrm{d}z$ and $b(x) = \nabla \ln \varphi_0(x) + (\int_{|z|\le 1} z\nu(z)\mathrm{d}z)x$ is the drift coefficient. Recall that the Hölder exponent of the pure jump Lévy term is equal to $1/(\delta_t \beta_\nu)$.

Define the point processes

$$\begin{aligned} p &= \{(s, z(s)); s \in D\} \\ \widetilde{p} &= \{(s, r(s)); s \in D\} \text{ with } r(s) = b(M_{s-} + z(s)) - b(M_{s-}), \end{aligned}$$

and the associated approximation rates δ_t (for p) and $\widetilde{\delta}_t$ (for $\widetilde{p}$) as in (3.6). The Hölder exponent of the drift term will depend on $\widetilde{p}$. Indeed, for the process $b(M_t) = G_t$ Lemma 3.1 implies $H_G(t) \le 1/(\beta_\nu \widetilde{\delta}_t)$. When b is only C^ε, the jump size $r(s) \le z(s)^\varepsilon$. Therefore, $H_G(t) \le \varepsilon/(\beta_\nu \delta_t)$ whenever $r(s) \asymp z(s)^\varepsilon$ occurs for infinitely many s tending to t.

On the other hand, in [1] it is shown that when the Hölder exponent of a Lévy process is less than $1/(2\beta_\nu)$ at some point t, or equivalently $\delta_t > 2$, the time t can not be an oscillating singularity in the sense that its primitive must have Hölder exponent $1 + 1/(\delta_t\beta_\nu)$ at time t. It is tempting to expect that the drift term here has Hölder exponent at most $1 + \varepsilon/(\delta_t\beta_\nu)$ as long as $\delta_t > 2$, for instance, equal to 3, and $r(s) \asymp z(s)^\varepsilon$ occurs for infinitely many s tending to t. If such a t exists, we get $1 + \varepsilon/(3\beta_\nu) < 1/(3\beta_\nu)$ for ε sufficiently small. This implies $H_M(t) \le 1 + \varepsilon/(3\beta_\nu)$, and changes the singularity sets $E_M(h)$ for $h \le 1 + \varepsilon/(3\beta_\nu)$.

Acknowledgments: JL thanks IHES, Bures-sur-Yvette, for a visiting fellowship, where part of this paper has been written. XY wishes to thank Prof. Yimin Xiao for stimulating discussions.

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